Differentiation of Vector-Valued Functions

In Sections 12.3–12.5, you will study several important applications involving the calculus of vector-valued functions. In preparation for that study, this section is devoted to the mechanics of differentiation and integration of vector-valued functions.

The definition of the derivative of a vector-valued function parallels that given for real-valued functions.

**Definition of the Derivative of a Vector-Valued Function**

The derivative of a vector-valued function \( \mathbf{r} \) is defined by

\[
\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}
\]

for all \( t \) for which the limit exists. If \( \mathbf{r}'(c) \) exists, then \( \mathbf{r} \) is differentiable at \( c \).

If \( \mathbf{r}'(c) \) exists for all \( c \) in an open interval \( I \), then \( \mathbf{r} \) is differentiable on the interval \( I \). Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

NOTE In addition to \( \mathbf{r}'(t) \), other notations for the derivative of a vector-valued function are

\[
D_t[\mathbf{r}(t)], \quad \frac{d}{dt}[\mathbf{r}(t)], \quad \text{and} \quad \frac{d\mathbf{r}}{dt}
\]

Differentiation of vector-valued functions can be done on a component-by-component basis. To see why this is true, consider the function given by

\[
\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}
\]

Applying the definition of the derivative produces the following.

\[
\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}
\]

\[
= \lim_{\Delta t \to 0} \frac{f(t + \Delta t)\mathbf{i} + g(t + \Delta t)\mathbf{j} - f(t)\mathbf{i} - g(t)\mathbf{j}}{\Delta t}
\]

\[
= \lim_{\Delta t \to 0} \left[ \frac{f(t + \Delta t) - f(t)}{\Delta t} \right] \mathbf{i} + \lim_{\Delta t \to 0} \left[ \frac{g(t + \Delta t) - g(t)}{\Delta t} \right] \mathbf{j}
\]

\[
= f'(t)\mathbf{i} + g'(t)\mathbf{j}
\]

This important result is listed in the theorem on the next page. Note that the derivative of the vector-valued function \( \mathbf{r} \) is itself a vector-valued function. You can see from Figure 12.8 that \( \mathbf{r}'(t) \) is a vector tangent to the curve given by \( \mathbf{r}(t) \) and pointing in the direction of increasing \( t \)-values.
THEOREM 12.1 Differentiation of Vector-Valued Functions

1. If \( r(t) = f(t)i + g(t)j \), where \( f \) and \( g \) are differentiable functions of \( t \), then
   \[
   r'(t) = f'(t)i + g'(t)j.
   \]
   Plane

2. If \( r(t) = f(t)i + g(t)j + h(t)k \), where \( f \), \( g \), and \( h \) are differentiable functions of \( t \), then
   \[
   r'(t) = f'(t)i + g'(t)j + h'(t)k.
   \]
   Space

EXAMPLE 1 Differentiation of Vector-Valued Functions

Find the derivative of each vector-valued function.

a. \( r(t) = t^2i - 4j \)
   b. \( r(t) = \frac{1}{t}i + \ln tj + e^{2t}k \)

Solution Differentiating on a component-by-component basis produces the following.

a. \[
   r'(t) = 2ti - 0j
   = 2ti
   \]
   Derivative

b. \[
   r'(t) = -\frac{1}{t^2}i + \frac{1}{t}j + 2e^{2t}k
   \]
   Derivative

Higher-order derivatives of vector-valued functions are obtained by successive differentiation of each component function.

EXAMPLE 2 Higher-Order Differentiation

For the vector-valued function given by \( r(t) = \cos ti + \sin tj + 2tk \), find each of the following.

a. \( r'(t) \)
   b. \( r''(t) \)
   c. \( r'(t) \cdot r''(t) \)
   d. \( r'(t) \times r''(t) \)

Solution

a. \[
   r'(t) = -\sin ti + \cos tj + 2k
   \]
   First derivative

b. \[
   r''(t) = -\cos ti - \sin tj + 0k
   = -\cos ti - \sin tj
   \]
   Second derivative

c. \[
   r'(t) \cdot r''(t) = \sin t \cos t - \sin t \cos t = 0
   \]
   Dot product

d. \[
   r'(t) \times r''(t) = \begin{vmatrix}
   i & j & k \\
   -\sin t & \cos t & 2 \\
   -\cos t & -\sin t & 0 
   \end{vmatrix}
   \]
   Cross product
   \[
   = \begin{vmatrix}
   \cos t & 2 & -\sin t \\
   -\sin t & -\cos t & 2 \\
   -\cos t & \sin t & 0 
   \end{vmatrix}i
   + \begin{vmatrix}
   \sin t & 2 & -\cos t \\
   -\cos t & -\sin t & 2 \\
   -\sin t & \cos t & 0 
   \end{vmatrix}j
   + \begin{vmatrix}
   \sin t & 2 & -\cos t \\
   -\cos t & -\sin t & 2 \\
   -\sin t & \cos t & 0 
   \end{vmatrix}k
   = 2 \sin ti - 2 \cos tj + k
   \]

Note that the dot product in part (c) is a real-valued function, not a vector-valued function.
The parametrization of the curve represented by the vector-valued function
\[ \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \]
is smooth on an open interval \( I \) if \( f', g', \) and \( h' \) are continuous on \( I \) and \( \mathbf{r}'(t) \neq \mathbf{0} \) for any value of \( t \) in the interval \( I \).

**Example 3** Finding Intervals on Which a Curve Is Smooth

Find the intervals on which the epicycloid \( C \) given by
\[ \mathbf{r}(t) = (5 \cos t - \cos 5t)\mathbf{i} + (5 \sin t - \sin 5t)\mathbf{j}, \quad 0 \leq t \leq 2\pi \]
is smooth.

**Solution** The derivative of \( \mathbf{r} \) is
\[ \mathbf{r}'(t) = (-5 \sin t + 5 \sin 5t)\mathbf{i} + (5 \cos t - 5 \cos 5t)\mathbf{j}. \]
In the interval \([0, 2\pi]\), the only values of \( t \) for which
\[ \mathbf{r}'(t) = 0\mathbf{i} + 0\mathbf{j} \]
are \( t = 0, \pi/2, \pi, 3\pi/2, \) and \( 2\pi \). Therefore, you can conclude that \( C \) is smooth in the intervals
\[ \left(0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right), \left(\pi, \frac{3\pi}{2}\right), \text{ and } \left(\frac{3\pi}{2}, 2\pi\right) \]
as shown in Figure 12.9.

**NOTE** In Figure 12.9, note that the curve is not smooth at points at which the curve makes abrupt changes in direction. Such points are called **cusps** or **nodes**.

Most of the differentiation rules in Chapter 2 have counterparts for vector-valued functions, and several are listed in the following theorem. Note that the theorem contains three versions of “product rules.” Property 3 gives the derivative of the product of a real-valued function \( f \) and a vector-valued function \( \mathbf{r} \). Property 4 gives the derivative of the dot product of two vector-valued functions, and Property 5 gives the derivative of the cross product of two vector-valued functions (in space). Note that Property 5 applies only to three-dimensional vector-valued functions, because the cross product is not defined for two-dimensional vectors.

**Theorem 12.2** Properties of the Derivative

Let \( \mathbf{r} \) and \( \mathbf{u} \) be differentiable vector-valued functions of \( t \), let \( f \) be a differentiable real-valued function of \( t \), and let \( c \) be a scalar.

1. \( D_t[cr(t)] = cr'(t) \)
2. \( D_t[\mathbf{r}(t) + \mathbf{u}(t)] = \mathbf{r}'(t) + \mathbf{u}'(t) \)
3. \( D_t[f(t)\mathbf{r}(t)] = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t) \)
4. \( D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t) \)
5. \( D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t) \)
6. \( D_t[f(\mathbf{r}(t))] = f'(\mathbf{r}(t))\mathbf{r}'(t) \)
7. If \( \mathbf{r}(t) \cdot \mathbf{r}(t) = c \), then \( \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \).
Proof  To prove Property 4, let 
\[ r(t) = f_1(t)i + g_1(t)j \quad \text{and} \quad u(t) = f_2(t)i + g_2(t)j \]
where \( f_1, f_2, g_1, \) and \( g_2 \) are differentiable functions of \( t \). Then,
\[ r(t) \cdot u(t) = f_1(t)f_2(t) + g_1(t)g_2(t) \]
and it follows that
\[
D_t[r(t) \cdot u(t)] = f_1(t)f_2'(t) + f_1'(t)f_2(t) + g_1(t)g_2'(t) + g_1'(t)g_2(t)
\]
\[ = [f_1(t)f_2'(t) + g_1(t)g_2'(t) + f_1'(t)f_2(t) + g_1'(t)g_2(t)]
\]
\[ = r(t) \cdot u'(t) + r'(t) \cdot u(t). \]
Proofs of the other properties are left as exercises (see Exercises 73–77 and Exercise 80).

**Example 4**  Using Properties of the Derivative

For the vector-valued functions given by
\[ r(t) = \frac{1}{t} i - j + \ln t k \quad \text{and} \quad u(t) = t^2 i - 2 tj + k \]
find
\[ a. \; D_t[r(t) \cdot u(t)] \quad \text{and} \quad b. \; D_t[u(t) \times u'(t)]. \]

**Solution**

**a.** Because \( r'(t) = -\frac{1}{t^2} i + \frac{1}{t} k \) and \( u'(t) = 2ti - 2j \), you have
\[
D_t[r(t) \cdot u(t)] = r(t) \cdot u'(t) + r'(t) \cdot u(t)
\]
\[ = \left( \frac{1}{t} i - j + \ln t k \right) \cdot (2ti - 2j) \]
\[ + \left( -\frac{1}{t^2} i + \frac{1}{t} k \right) \cdot (t^2 i - 2 tj + k) \]
\[ = 2 + 2 + (-1) + \frac{1}{t} \]
\[ = 3 + \frac{1}{t}. \]

**b.** Because \( u'(t) = 2ti - 2j \) and \( u''(t) = 2i \), you have
\[
D_t[u(t) \times u'(t)] = [u(t) \times u''(t)] + [u'(t) \times u'(t)]
\]
\[ = \begin{vmatrix} i & j & k \\ t^2 & -2t & 1 \\ 2 & 0 & 0 \end{vmatrix} + 0
\]
\[ = \begin{vmatrix} -2t & 1 & \frac{1}{2} \\ 0 & 2 & 0 \end{vmatrix} + \begin{vmatrix} t^2 & -2t & 1 \\ 0 & 2 & 0 \end{vmatrix} k
\]
\[ = 6i - (-2)j + 4tk
\]
\[ = 2j + 4tk. \]

**NOTE** Try reworking parts (a) and (b) in Example 4 by first forming the dot and cross products and then differentiating to see that you obtain the same results.
Integration of Vector-Valued Functions

The following definition is a rational consequence of the definition of the derivative of a vector-valued function.

**Definition of Integration of Vector-Valued Functions**

1. If \( \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}, \) where \( f \) and \( g \) are continuous on \([a, b]\), then the **indefinite integral** (antiderivative) of \( \mathbf{r} \) is

\[
\int \mathbf{r}(t) \, dt = \left[ \int f(t) \, dt \right] \mathbf{i} + \left[ \int g(t) \, dt \right] \mathbf{j}
\]

and its **definite integral** over the interval \( a \leq t \leq b \) is

\[
\int_a^b \mathbf{r}(t) \, dt = \left[ \int_a^b f(t) \, dt \right] \mathbf{i} + \left[ \int_a^b g(t) \, dt \right] \mathbf{j}.
\]

2. If \( \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}, \) where \( f, g, \) and \( h \) are continuous on \([a, b]\), then the **indefinite integral** (antiderivative) of \( \mathbf{r} \) is

\[
\int \mathbf{r}(t) \, dt = \left[ \int f(t) \, dt \right] \mathbf{i} + \left[ \int g(t) \, dt \right] \mathbf{j} + \left[ \int h(t) \, dt \right] \mathbf{k}
\]

and its **definite integral** over the interval \( a \leq t \leq b \) is

\[
\int_a^b \mathbf{r}(t) \, dt = \left[ \int_a^b f(t) \, dt \right] \mathbf{i} + \left[ \int_a^b g(t) \, dt \right] \mathbf{j} + \left[ \int_a^b h(t) \, dt \right] \mathbf{k}.
\]

The antiderivative of a vector-valued function is a family of vector-valued functions all differing by a constant vector \( \mathbf{C} \). For instance, if \( \mathbf{r}(t) \) is a three-dimensional vector-valued function, then for the indefinite integral \( \int \mathbf{r}(t) \, dt \), you obtain three constants of integration

\[
\int f(t) \, dt = F(t) + C_1, \quad \int g(t) \, dt = G(t) + C_2, \quad \int h(t) \, dt = H(t) + C_3
\]

where \( F'(t) = f(t), \ G'(t) = g(t), \) and \( H'(t) = h(t). \) These three scalar constants produce one vector constant of integration,

\[
\int \mathbf{r}(t) \, dt = \left[ F(t) + C_1 \right] \mathbf{i} + \left[ G(t) + C_2 \right] \mathbf{j} + \left[ H(t) + C_3 \right] \mathbf{k}
\]

\[
= \left[ F(t) \mathbf{i} + G(t) \mathbf{j} + H(t) \mathbf{k} \right] + \left[ C_1 \mathbf{i} + C_2 \mathbf{j} + C_3 \mathbf{k} \right]
\]

\[
= \mathbf{R}(t) + \mathbf{C}
\]

where \( \mathbf{R}'(t) = \mathbf{r}(t). \)

**Example 5** Integrating a Vector-Valued Function

Find the indefinite integral

\[
\int (t \mathbf{i} + 3\mathbf{j}) \, dt.
\]

**Solution** Integrating on a component-by-component basis produces

\[
\int (t \mathbf{i} + 3\mathbf{j}) \, dt = \frac{t^2}{2} \mathbf{i} + 3t \mathbf{j} + \mathbf{C}.
\]
Example 6 shows how to evaluate the definite integral of a vector-valued function.

**EXAMPLE 6  Definite Integral of a Vector-Valued Function**

Evaluate the integral

\[ \int_0^1 \mathbf{r}(t) \, dt = \int_0^1 \left( \sqrt{t} \mathbf{i} + \frac{1}{t + 1} \mathbf{j} + e^{-t} \mathbf{k} \right) \, dt. \]

**Solution**

\[
\begin{align*}
\int_0^1 \mathbf{r}(t) \, dt &= \left( \int_0^1 t^{1/3} \, dt \right) \mathbf{i} + \left( \int_0^1 \frac{1}{t + 1} \, dt \right) \mathbf{j} + \left( \int_0^1 e^{-t} \, dt \right) \mathbf{k} \\
&= \left[ \left( \frac{3}{4} \right) t^{4/3} \right]_0^1 \mathbf{i} + \left[ \ln |t + 1| \right]_0^1 \mathbf{j} + \left[ -e^{-t} \right]_0^1 \mathbf{k} \\
&= \frac{3}{4} \mathbf{i} + (\ln 2) \mathbf{j} + \left( 1 - \frac{1}{e} \right) \mathbf{k}
\end{align*}
\]

As with real-valued functions, you can narrow the family of antiderivatives of a vector-valued function \( \mathbf{r}' \) down to a single antiderivative by imposing an initial condition on the vector-valued function \( \mathbf{r} \). This is demonstrated in the next example.

**EXAMPLE 7  The Antiderivative of a Vector-Valued Function**

Find the antiderivative of

\[ \mathbf{r}'(t) = \cos 2t \mathbf{i} + 2 \sin t \mathbf{j} + \frac{1}{1 + t^2} \mathbf{k} \]

that satisfies the initial condition \( \mathbf{r}(0) = 3 \mathbf{i} - 2 \mathbf{j} + \mathbf{k} \).

**Solution**

\[
\begin{align*}
\mathbf{r}(t) &= \int \mathbf{r}'(t) \, dt \\
&= \left( \int \cos 2t \, dt \right) \mathbf{i} + \left( \int -2 \sin t \, dt \right) \mathbf{j} + \left( \int \frac{1}{1 + t^2} \, dt \right) \mathbf{k} \\
&= \left( \frac{1}{2} \sin 2t + C_1 \right) \mathbf{i} + \left( 2 \cos t + C_2 \right) \mathbf{j} + \left( \arctan t + C_3 \right) \mathbf{k}
\end{align*}
\]

Letting \( t = 0 \) and using the fact that \( \mathbf{r}(0) = 3 \mathbf{i} - 2 \mathbf{j} + \mathbf{k} \), you have

\[
\begin{align*}
\mathbf{r}(0) &= (0 + C_1) \mathbf{i} + (2 + C_2) \mathbf{j} + (0 + C_3) \mathbf{k} \\
&= 3 \mathbf{i} - 2 \mathbf{j} + \mathbf{k}.
\end{align*}
\]

Equating corresponding components produces

\[
C_1 = 3, \quad 2 + C_2 = -2, \quad \text{and} \quad C_3 = 1.
\]

So, the antiderivative that satisfies the given initial condition is

\[ \mathbf{r}(t) = \left( \frac{1}{2} \sin 2t + 3 \right) \mathbf{i} + (2 \cos t - 4) \mathbf{j} + (\arctan t + 1) \mathbf{k}. \]
In Exercises 1–6, sketch the plane curve represented by the vector-valued function, and sketch the vectors \( \mathbf{r}(t_0) \) and \( \mathbf{r}'(t_0) \) for the given value of \( t_0 \). Position the vectors such that the initial point of \( \mathbf{r}(t_0) \) is at the origin and the initial point of \( \mathbf{r}'(t_0) \) is at the terminal point of \( \mathbf{r}(t_0) \). What is the relationship between \( \mathbf{r}'(t_0) \) and the curve?

1. \( \mathbf{r}(t) = t^2 \mathbf{i} + t \mathbf{j}, \quad t_0 = 2 \)
2. \( \mathbf{r}(t) = \mathbf{i} + t^2 \mathbf{j}, \quad t_0 = 1 \)
3. \( \mathbf{r}(t) = t^2 \mathbf{i} + \frac{1}{t} \mathbf{j}, \quad t_0 = 2 \)
4. \( \mathbf{r}(t) = (1 + t) \mathbf{i} + t^2 \mathbf{j}, \quad t_0 = 1 \)
5. \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}, \quad t_0 = \frac{\pi}{2} \)
6. \( \mathbf{r}(t) = e^t \mathbf{i} + e^{2t} \mathbf{j}, \quad t_0 = 0 \)

7. **Investigation** Consider the vector-valued function \( \mathbf{r}(t) = \mathbf{i} + t \mathbf{j} \).
   
   (a) Sketch the graph of \( \mathbf{r}(t) \). Use a graphing utility to verify your graph.
   
   (b) Sketch the vectors \( \mathbf{r}(1/4), \mathbf{r}(1/2), \text{ and } \mathbf{r}(1/2) - \mathbf{r}(1/4) \) on the graph in part (a).
   
   (c) Compare the vector \( \mathbf{r}'(1/4) \) with the vector \( \mathbf{r}(1/2) - \mathbf{r}(1/4) \).

8. **Investigation** Consider the vector-valued function \( \mathbf{r}(t) = \mathbf{i} + (4 - t^2) \mathbf{j} \).
   
   (a) Sketch the graph of \( \mathbf{r}(t) \). Use a graphing utility to verify your graph.
   
   (b) Sketch the vectors \( \mathbf{r}(1), \mathbf{r}(1.25), \text{ and } \mathbf{r}(1.25) - \mathbf{r}(1) \) on the graph in part (a).
   
   (c) Compare the vector \( \mathbf{r}'(1) \) with the vector \( \frac{\mathbf{r}(1.25) - \mathbf{r}(1)}{1.25 - 1} \).

In Exercises 9 and 10, (a) sketch the space curve represented by the vector-valued function, and (b) sketch the vectors \( \mathbf{r}(t_0) \) and \( \mathbf{r}'(t_0) \) for the given value of \( t_0 \).

9. \( \mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + t \mathbf{k}, \quad t_0 = \frac{3\pi}{2} \)
10. \( \mathbf{r}(t) = \mathbf{i} + t^2 \mathbf{j} + \frac{2}{t} \mathbf{k}, \quad t_0 = 2 \)

In Exercises 11–18, find \( \mathbf{r}'(t) \).

11. \( \mathbf{r}(t) = 6 \mathbf{i} - 7t \mathbf{j} + t^2 \mathbf{k} \)
12. \( \mathbf{r}(t) = \frac{1}{t} \mathbf{i} + 16t \mathbf{j} + \frac{t^2}{2} \mathbf{k} \)
13. \( \mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j} + \mathbf{k} \)
14. \( \mathbf{r}(t) = 4\sqrt{t} \mathbf{i} + t^2 \sqrt{t} \mathbf{j} + \ln t \mathbf{k} \)
15. \( \mathbf{r}(t) = e^{-t} \mathbf{i} + 4 \mathbf{j} \)
16. \( \mathbf{r}(t) = (\sin t - t \cos t, \cos t + t \sin t, t^2) \)
17. \( \mathbf{r}(t) = (t \sin t, t \cos t, t) \)
18. \( \mathbf{r}(t) = (\arcsin t, \arccos t, 0) \)

In Exercises 19–26, find (a) \( \mathbf{r}''(t) \) and (b) \( \mathbf{r}'(t) \cdot \mathbf{r}''(t) \).

19. \( \mathbf{r}(t) = t^2 \mathbf{i} + \frac{3}{2} t^2 \mathbf{j} \)
20. \( \mathbf{r}(t) = (t^2 + t) \mathbf{i} + (t^2 - t) \mathbf{j} \)
21. \( \mathbf{r}(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j} \)
22. \( \mathbf{r}(t) = 8 \cos t \mathbf{i} + 3 \sin t \mathbf{j} \)
23. \( \mathbf{r}(t) = \frac{1}{2} t^2 \mathbf{i} - t \mathbf{j} + \frac{1}{2} t \mathbf{k} \)
24. \( \mathbf{r}(t) = t \mathbf{i} + (2t + 3) \mathbf{j} + (3t - 5) \mathbf{k} \)
25. \( \mathbf{r}(t) = (\cos t + t \sin t, \sin t - t \cos t, t) \)
26. \( \mathbf{r}(t) = (e^{-t}, t^2, \tan t) \)

In Exercises 27 and 28, a vector-valued function and its graph are given. The graph also shows the unit vectors \( \mathbf{r}'(t_0) \parallel \mathbf{r}''(t_0) \) and \( \mathbf{r}''(t_0) \parallel \mathbf{r}''(t_0) \). Find these two unit vectors and identify them on the graph.

27. \( \mathbf{r}(t) = \cos(\pi t) \mathbf{i} + \sin(\pi t) \mathbf{j} + t^2 \mathbf{k}, \quad t_0 = -\frac{1}{4} \)
28. \( \mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + e^{3t} \mathbf{k}, \quad t_0 = \frac{1}{2} \)

In Exercises 29–38, find the open interval(s) on which the curve given by the vector-valued function is smooth.

29. \( \mathbf{r}(t) = t^2 \mathbf{i} + t^2 \mathbf{j} \)
30. \( \mathbf{r}(t) = \frac{1}{t - 1} \mathbf{i} + 3t \mathbf{j} \)
31. \( \mathbf{r}(t) = 2 \cos^3 t \mathbf{i} + 3 \sin^3 t \mathbf{j} \)
32. \( \mathbf{r}(t) = (\theta + \sin \theta) \mathbf{i} + (1 - \cos \theta) \mathbf{j} \)
33. \( \mathbf{r}(t) = (\theta - 2 \sin \theta) \mathbf{i} + (1 - 2 \cos \theta) \mathbf{j} \)
34. \( \mathbf{r}(t) = \frac{2t}{8 + t^4} + \frac{2t^2}{8 + t^4} \mathbf{j} \)
35. \( \mathbf{r}(t) = (t - 1) \mathbf{i} + \frac{1}{t} \mathbf{j} - t \mathbf{k} \)
36. \( \mathbf{r}(t) = e^{-t} \mathbf{i} - e^{-t} \mathbf{j} + 3t \mathbf{k} \)
37. \( \mathbf{r}(t) = t \mathbf{i} - 3(t^2 + \tan t) \mathbf{k} \)
38. \( \mathbf{r}(t) = \sqrt{t} \mathbf{i} + (t^2 - 1) \mathbf{j} + \frac{1}{2} \mathbf{k} \)

In Exercises 39 and 40, use the properties of the derivative to find the following.

(a) \( \mathbf{r}'(t) \quad (b) \mathbf{r}''(t) \quad (c) D_1[\mathbf{r}(t) \cdot \mathbf{u}(t)] \)
(d) \( \mathbf{D}_2[\mathbf{r}(t) - \mathbf{u}(t)] \quad (e) \mathbf{D}_1[\mathbf{r}(t) \times \mathbf{u}(t)] \quad (f) \mathbf{D}_1[I[\mathbf{r}(t)]], \quad t > 0 \)
39. \( \mathbf{r}(t) = ti + 3tj + r^2k, \quad \mathbf{u}(t) = 4ti + r^2j + r^2k \)
40. \( \mathbf{r}(t) = ti + 2sintj + 2cosrk, \quad \mathbf{u}(t) = t^2i + 2sintj + 2cosrk \)
In Exercises 41 and 42, find (a) $D_t[r(t) \cdot u(t)]$ and (b) $D_t[r(t) \times u(t)]$ by differentiating the product, then applying the properties of Theorem 12.2.

41. $\mathbf{r}(t) = \hat{i} + 2\hat{j} + t\hat{k}$, $\mathbf{u}(t) = r\hat{k}$
42. $\mathbf{r}(t) = \cos r\hat{i} + \sin t\hat{j} + t\hat{k}$, $\mathbf{u}(t) = \hat{j} + t\hat{k}$

In Exercises 43 and 44, find the angle $\theta$ between $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ as a function of $t$. Use a graphing utility to graph $\theta(t)$. Use the graph to find any extrema of the function. Find any values of $t$ at which the vectors are orthogonal.

43. $\mathbf{r}(t) = 3 \sin t\hat{i} + 4 \cos t\hat{j}$
44. $\mathbf{r}(t) = t^2\hat{i} + t\hat{j}$

In Exercises 45–48, use the definition of the derivative to find $\mathbf{r}'(t)$.

45. $\mathbf{r}(t) = (3t + 2)\hat{i} + (1 - t^2)\hat{j}$
46. $\mathbf{r}(t) = \sqrt{7} \hat{i} + \frac{3}{7} \hat{j} + 2t\hat{k}$
47. $\mathbf{r}(t) = (t^2, 0, 2t)$
48. $\mathbf{r}(t) = (0, \sin t, 4t)$

In Exercises 49–56, find the indefinite integral.

49. $\int (2t + j + k) \, dt$
50. $\int (4t^2 \hat{i} + 6tj - 4\sqrt{7}k) \, dt$
51. $\int \left[ \frac{1}{7} \hat{i} + j - t^{3/2}k \right] \, dt$
52. $\int \left[ \ln r \hat{i} + \frac{1}{r} \hat{j} + k \right] \, dt$
53. $\int \left[ (2t^{3/2} \hat{i} + 4t^2 \hat{j} + 3\sqrt{7}k) \right] \, dt$
54. $\int \left[ \sec^2 t \hat{i} + \frac{1}{1 + t^2} \hat{j} \right] \, dt$
55. $\int \left[ \sec t \tan t \hat{i} + (\tan t) \hat{j} + (2 \sin t \cos t) \hat{k} \right] \, dt$
56. $\int \left[ e^{-t} \sin t \hat{i} + e^{-t} \cos t \hat{j} \right] \, dt$

In Exercises 57–62, evaluate the definite integral.

57. $\int_0^\pi (8\hat{i} + j - k) \, dt$
58. $\int_0^\pi (\hat{i} + t\hat{j} + \sqrt{7}k) \, dt$
59. $\int_0^{\pi/2} \left[ (a \cos t) \hat{i} + (a \sin t) \hat{j} + k \right] \, dt$
60. $\int_0^{\pi/4} \left[ (\sec t \tan t) \hat{i} + (\tan t) \hat{j} + (2 \sin t \cos t) \hat{k} \right] \, dt$
61. $\int_0^2 (t\hat{i} + e^{-t} \hat{j} - te^t \hat{k}) \, dt$
62. $\int_0^1 \| \hat{i} + t^2 \hat{j} \| \, dt$

In Exercises 63–68, find $\mathbf{r}(t)$ for the given conditions.

63. $\mathbf{r}'(t) = 4e^{3t} + 3e^t\hat{j}$, $\mathbf{r}(0) = 2\hat{i}$
64. $\mathbf{r}'(t) = 3t^{3/2} + 6\sqrt{7}\hat{k}$, $\mathbf{r}(0) = \hat{i} + 2\hat{j}$
65. $\mathbf{r}'(t) = -32\hat{j}$, $\mathbf{r}'(0) = 600\sqrt{5}\hat{i} + 600\hat{j}$, $\mathbf{r}(0) = 0$
66. $\mathbf{r}'(t) = -4 \cos 3t \hat{i} - 3 \sin t \hat{j}$, $\mathbf{r}'(0) = 3\hat{k}$, $\mathbf{r}(0) = 0$
67. $\mathbf{r}'(t) = te^{-t}\hat{i} - e^{-t} \hat{j} + \hat{k}$, $\mathbf{r}(0) = \frac{1}{2}\hat{i} - \hat{j} + \hat{k}$
68. $\mathbf{r}'(t) = \frac{1}{1 + t^2} \hat{i} + \frac{1}{t^2} \hat{j} + \frac{1}{t} \hat{k}$, $\mathbf{r}(1) = 2\hat{i}$

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**Writing About Concepts**

69. State the definition of the derivative of a vector-valued function. Describe how to find the derivative of a vector-valued function and give its geometric interpretation.
70. How do you find the integral of a vector-valued function?
71. The three components of the derivative of the vector-valued function $\mathbf{u}$ are positive at $t = t_0$. Describe the behavior of $\mathbf{u}$ at $t = t_0$.
72. The $z$-component of the derivative of the vector-valued function $\mathbf{u}$ is 0 for $t$ in the domain of the function. What does this information imply about the graph of $\mathbf{u}$?

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In Exercises 73–80, prove the property. In each case, assume $\mathbf{r}$, $\mathbf{u}$, and $\mathbf{v}$ are differentiable vector-valued functions of $t$, $f$ is a differentiable real-valued function of $t$, and $c$ is a scalar.

73. $D_c[\mathbf{r}(t)] = c\mathbf{r}'(t)$
74. $D_c[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
75. $D_c[f(t)\mathbf{r}(t)] = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$
76. $D_c[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}'(t) \times \mathbf{u}'(t)$
77. $D_c[f(\mathbf{r}(t))] = f'(\mathbf{r}(t))\mathbf{r}'(t)$
78. $D_c[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}'(t)$
79. $D_c[\mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)]] = \mathbf{r}'(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}'(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}'(t)]$
80. If $\mathbf{r}(t) \cdot \mathbf{r}'(t)$ is a constant, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

**81. Particle Motion** A particle moves in the $xy$-plane along the curve represented by the vector-valued function $\mathbf{r}(t) = (1 - \sin t)\hat{i} + (1 - \cos t)\hat{j}$.

(a) Use a graphing utility to graph $\mathbf{r}$. Describe the curve.
(b) Find the minimum and maximum values of $\| \mathbf{r} \|$ and $\| \mathbf{r}' \|$.

**82. Particle Motion** A particle moves in the $yz$-plane along the curve represented by the vector-valued function $\mathbf{r}(t) = (2 \cos t)\hat{j} + (3 \sin t)\hat{k}$.

(a) Describe the curve.
(b) Find the minimum and maximum values of $\| \mathbf{r} \|$ and $\| \mathbf{r}' \|$.

**True or False?** In Exercises 83–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

83. If a particle moves along a sphere centered at the origin, then its derivative vector is always tangent to the sphere.
84. The definite integral of a vector-valued function is a real number.
85. $\frac{d}{dt}[\|\mathbf{r}(t)\|] = \|\mathbf{r}'(t)\|$
86. If $\mathbf{r}$ and $\mathbf{u}$ are differentiable vector-valued functions of $t$, then $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}'(t) \cdot \mathbf{u}(t) + \mathbf{r}(t) \cdot \mathbf{u}'(t)$.
87. Consider the vector-valued function $\mathbf{r}(t) = (e^t \sin t)\hat{i} + (e^t \cos t)\hat{j}$.

Show that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are always perpendicular to each other.