Section 12.4

Tangent Vectors and Normal Vectors

- Find a unit tangent vector at a point on a space curve.
- Find the tangential and normal components of acceleration.

Tangent Vectors and Normal Vectors

In the preceding section, you learned that the velocity vector points in the direction of motion. This observation leads to the following definition, which applies to any smooth curve—not just to those for which the parameter represents time.

Recall that a curve is smooth on an interval if is continuous and nonzero on the interval. So, “smoothness” is sufficient to guarantee that a curve has a unit tangent vector.

**EXAMPLE 1  Finding the Unit Tangent Vector**

Find the unit tangent vector to the curve given by
\[ r(t) = ti + t^2j \]
when \( t = 1 \).

**Solution**  The derivative of \( r(t) \) is
\[ r'(t) = i + 2tj. \]

So, the unit tangent vector is
\[ T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{1 + 4t^2}}(i + 2tj). \]

When \( t = 1 \), the unit tangent vector is
\[ T(1) = \frac{1}{\sqrt{5}}(i + 2j) \]
as shown in Figure 12.19.

**NOTE**  In Example 1, note that the direction of the unit tangent vector depends on the orientation of the curve. For instance, if the parabola in Figure 12.19 were given by
\[ r(t) = -(t - 2)i + (t - 2)^2j, \]
\( T(1) \) would still represent the unit tangent vector at the point \((1, 1)\), but it would point in the opposite direction. Try verifying this.
The **tangent line to a curve** at a point is the line passing through the point and parallel to the unit tangent vector. In Example 2, the unit tangent vector is used to find the tangent line at a point on a helix.

**EXAMPLE 2** Finding the Tangent Line at a Point on a Curve

Find \( T(t) \) and then find a set of parametric equations for the tangent line to the helix given by

\[
\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + t \mathbf{k}
\]

at the point corresponding to \( t = \pi/4 \).

**Solution** The derivative of \( \mathbf{r}(t) \) is \( \mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \mathbf{k} \), which implies that \( \| \mathbf{r}'(t) \| = \sqrt{4 \sin^2 t + 4 \cos^2 t + 1} = \sqrt{5} \). Therefore, the unit tangent vector is

\[
T(t) = \frac{\mathbf{r}'(t)}{\| \mathbf{r}'(t) \|} = \frac{1}{\sqrt{5}}(-2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \mathbf{k}).
\]

When \( t = \pi/4 \), the unit tangent vector is

\[
T\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{5}}\left(-2 \frac{\sqrt{2}}{2} \mathbf{i} + 2 \frac{\sqrt{2}}{2} \mathbf{j} + \mathbf{k}\right)
\]

\[
= \frac{1}{\sqrt{5}}(-\sqrt{2} \mathbf{i} + \sqrt{2} \mathbf{j} + \mathbf{k}).
\]

Using the direction numbers \( a = -\sqrt{2} \), \( b = \sqrt{2} \), and \( c = 1 \), and the point \((x, y, z) = (\sqrt{2}, \sqrt{2}, \pi/4)\), you can obtain the following parametric equations (given with parameter \( s \)).

\[
x = x_1 + as = \sqrt{2} - \sqrt{2}s
\]

\[
y = y_1 + bs = \sqrt{2} + \sqrt{2}s
\]

\[
z = z_1 + cs = \frac{\pi}{4} + s
\]

This tangent line is shown in Figure 12.20.

In Example 2, there are infinitely many vectors that are orthogonal to the tangent vector \( T(t) \). One of these is the vector \( T'(t) \). This follows from Property 7 of Theorem 12.2. That is,

\[
T(t) \cdot T(t) = \| T(t) \|^2 = 1 \quad \Rightarrow \quad T(t) \cdot T'(t) = 0.
\]

By normalizing the vector \( T'(t) \), you obtain a special vector called the **principal unit normal vector**, as indicated in the following definition.

**Definition of Principal Unit Normal Vector**

Let \( C \) be a smooth curve represented by \( \mathbf{r} \) on an open interval \( I \). If \( T'(t) \neq 0 \), then the **principal unit normal vector** at \( t \) is defined to be

\[
\mathbf{N}(t) = \frac{T'(t)}{\| T'(t) \|}.
\]
**EXAMPLE 3  Finding the Principal Unit Normal Vector**

Find $N(t)$ and $N(1)$ for the curve represented by

$$r(t) = 3t \mathbf{i} + 2t^2 \mathbf{j}.$$  

**Solution**  By differentiating, you obtain

$$r'(t) = 3 \mathbf{i} + 4t \mathbf{j} \quad \text{and} \quad \|r'(t)\| = \sqrt{9 + 16t^2}$$

which implies that the unit tangent vector is

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{9 + 16t^2}}(3 \mathbf{i} + 4t \mathbf{j}). \quad \text{Unit tangent vector}$$

Using Theorem 12.2, differentiate $T(t)$ with respect to $t$ to obtain

$$T'(t) = \frac{1}{\sqrt{9 + 16t^2}}(4j) - \frac{16t}{(9 + 16t^2)^{3/2}}(3 \mathbf{i} + 4t \mathbf{j})$$

$$= \frac{12}{(9 + 16t^2)^{3/2}}(-4t \mathbf{i} + 3 \mathbf{j})$$

$$\|T'(t)\| = 12 \sqrt{\frac{9 + 16t^2}{(9 + 16t^2)^3}} = \frac{12}{9 + 16t^2}$$

Therefore, the principal unit normal vector is

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{1}{\sqrt{9 + 16t^2}}(-4t \mathbf{i} + 3 \mathbf{j}). \quad \text{Principal unit normal vector}$$

When $t = 1$, the principal unit normal vector is

$$N(1) = \frac{1}{5}(-4 \mathbf{i} + 3 \mathbf{j})$$

as shown in Figure 12.21.

The principal unit normal vector can be difficult to evaluate algebraically. For plane curves, you can simplify the algebra by finding

$$T(t) = x(t) \mathbf{i} + y(t) \mathbf{j} \quad \text{Unit tangent vector}$$

and observing that $N(t)$ must be either

$$N_1(t) = y(t) \mathbf{i} - x(t) \mathbf{j} \quad \text{or} \quad N_2(t) = -y(t) \mathbf{i} + x(t) \mathbf{j}.$$  

Because $\sqrt{(x(t))^2 + (y(t))^2} = 1$, it follows that both $N_1(t)$ and $N_2(t)$ are unit normal vectors. The principal unit normal vector $N$ is the one that points toward the concave side of the curve, as shown in Figure 12.21 (see Exercise 86). This also holds for curves in space. That is, for an object moving along a curve $C$ in space, the vector $T(t)$ points in the direction the object is moving, whereas the vector $N(t)$ is orthogonal to $T(t)$ and points in the direction in which the object is turning, as shown in Figure 12.22.
EXAMPLE 4  Finding the Principal Unit Normal Vector

Find the principal unit normal vector for the helix given by
\[ r(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + t \mathbf{k}. \]

**Solution**  From Example 2, you know that the unit tangent vector is
\[ T(t) = \left\langle \frac{1}{\sqrt{5}} (-2 \sin t \mathbf{i} + 2 \cos t \mathbf{j}) \right\rangle. \]

So, \( T'(t) \) is given by
\[ T'(t) = \frac{1}{\sqrt{5}} (-2 \cos t \mathbf{i} - 2 \sin t \mathbf{j}). \]

Because \( \|T'(t)\| = 2/\sqrt{5} \), it follows that the principal unit normal vector is
\[ N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{1}{2} (-2 \cos t \mathbf{i} - 2 \sin t \mathbf{j}) \]
\[ = -\cos t \mathbf{i} - \sin t \mathbf{j}. \]

Note that this vector is horizontal and points toward the \( z \)-axis, as shown in Figure 12.23.

**Tangential and Normal Components of Acceleration**

Let’s return to the problem of describing the motion of an object along a curve. In the preceding section, you saw that for an object traveling at a **constant speed**, the velocity and acceleration vectors are perpendicular. This seems reasonable, because the speed would not be constant if any acceleration were acting in the direction of motion. You can verify this observation by noting that
\[ \mathbf{r}''(t) \cdot \mathbf{r}'(t) = 0 \]
if \( \|\mathbf{r}'(t)\| \) is a constant. (See Property 7 of Theorem 12.2.)

However, for an object traveling at a **variable speed**, the velocity and acceleration vectors are not necessarily perpendicular. For instance, you saw that the acceleration vector for a projectile always points down, regardless of the direction of motion.

In general, part of the acceleration (the tangential component) acts in the line of motion, and part (the normal component) acts perpendicular to the line of motion. In order to determine these two components, you can use the unit vectors \( \mathbf{T}(t) \) and \( \mathbf{N}(t) \), which serve in much the same way as do \( \mathbf{i} \) and \( \mathbf{j} \) in representing vectors in the plane. The following theorem states that the acceleration vector lies in the plane determined by \( \mathbf{T}(t) \) and \( \mathbf{N}(t) \).

**THEOREM 12.4  Acceleration Vector**

If \( \mathbf{r}(t) \) is the position vector for a smooth curve \( C \) and \( \mathbf{N}(t) \) exists, then the acceleration vector \( \mathbf{a}(t) \) lies in the plane determined by \( \mathbf{T}(t) \) and \( \mathbf{N}(t) \).
Proof  To simplify the notation, write $T$ for $T(t)$, $T'$ for $T'(t)$, and so on. Because $T = r'/\|r'\| = v/\|v\|$, it follows that

$$v = \|v\|T.$$

By differentiating, you obtain

$$a = v' = D_1[\|v\|]T + \|v\|T' = D_1[\|v\|]T + \|v\|T\left(\frac{T'}{\|T'\|}\right) = D_1[\|v\|]T + \|v\|\|T'\|N = T/\|T'\|.$$

Because $a$ is written as a linear combination of $T$ and $N$, it follows that $a$ lies in the plane determined by $T$ and $N$.

The coefficients of $T$ and $N$ in the proof of Theorem 12.4 are called the tangential and normal components of acceleration and are denoted by $a_T = D_1[\|v\|]$ and $a_N = \|v\||T'|$. So, you can write

$$a(t) = a_TT(t) + a_NT(t).$$

The following theorem gives some convenient formulas for $a_N$ and $a_T$.

**THEOREM 12.5  Tangential and Normal Components of Acceleration**

If $r(t)$ is the position vector for a smooth curve $C$ [for which $N(t)$ exists], then the tangential and normal components of acceleration are as follows.

$$a_T = D_1[\|v\|] = a \cdot T = \frac{v \cdot a}{\|v\|}$$

$$a_N = \|v\||T'| = a \cdot N = \frac{\|v \times a\|}{\|v\|} = \sqrt{\|a\|^2 - a_T^2}$$

Note that $a_N \geq 0$. The normal component of acceleration is also called the centripetal component of acceleration.

Proof  Note that $a$ lies in the plane of $T$ and $N$. So, you can use Figure 12.24 to conclude that, for any time $t$, the component of the projection of the acceleration vector onto $T$ is given by $a_T = a \cdot T$, and onto $N$ is given by $a_N = a \cdot N$. Moreover, because $a = v'$ and $T = v/\|v\|$, you have

$$a_T = a \cdot T = T \cdot a = \frac{v}{\|v\|} \cdot a = \frac{v \cdot a}{\|v\|}.$$

In Exercises 88 and 89, you are asked to prove the other parts of the theorem.

NOTE  The formulas from Theorem 12.5, together with several other formulas from this chapter, are summarized on page 875.
EXAMPLE 5  Tangential and Normal Components of Acceleration

Find the tangential and normal components of acceleration for the position vector given by \( \mathbf{r}(t) = 3t \mathbf{i} - t^2 \mathbf{j} + t^3 \mathbf{k} \).

**Solution**  Begin by finding the velocity, speed, and acceleration.

\[
\mathbf{v}(t) = \mathbf{r}'(t) = 3 \mathbf{i} - 2t \mathbf{k}
\]

\[
\|\mathbf{v}(t)\| = \sqrt{9 + 1 + 4t^2} = \sqrt{10 + 4t^2}
\]

\[
\mathbf{a}(t) = \mathbf{r}''(t) = 2 \mathbf{k}
\]

By Theorem 12.5, the tangential component of acceleration is

\[
a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{4t}{\sqrt{10 + 4t^2}}
\]

and because

\[
\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = -2\mathbf{i} - 6\mathbf{j}
\]

the normal component of acceleration is

\[
a_N = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \frac{\sqrt{4 + 36}}{\sqrt{10 + 4t^2}} = \frac{2\sqrt{10}}{\sqrt{10 + 4t^2}}
\]

NOTE  In Example 5, you could have used the alternative formula for \( a_N \) as follows.

\[
a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} = \sqrt{(2)^2 - \frac{16t^2}{10 + 4t^2}} = \frac{2\sqrt{10}}{\sqrt{10 + 4t^2}}
\]

EXAMPLE 6  Finding \( a_T \) and \( a_N \) for a Circular Helix

Find the tangential and normal components of acceleration for the helix given by \( \mathbf{r}(t) = b \cos t \mathbf{i} + b \sin t \mathbf{j} + ct \mathbf{k}, b > 0 \).

**Solution**

\[
\mathbf{v}(t) = \mathbf{r}'(t) = -b \sin t \mathbf{i} + b \cos t \mathbf{j} + c \mathbf{k}
\]

\[
\|\mathbf{v}(t)\| = \sqrt{b^2 \sin^2 t + b^2 \cos^2 t + c^2} = \sqrt{b^2 + c^2}
\]

\[
\mathbf{a}(t) = \mathbf{r}''(t) = -b \cos t \mathbf{i} - b \sin t \mathbf{j}
\]

By Theorem 12.5, the tangential component of acceleration is

\[
a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{b^2 \sin t \cos t - b^2 \sin t \cos t + 0}{\sqrt{b^2 + c^2}} = 0.
\]

Tangential component of acceleration

Moreover, because \( \|\mathbf{a}\| = \sqrt{b^2 \cos^2 t + b^2 \sin^2 t} = b \), you can use the alternative formula for the normal component of acceleration to obtain

\[
a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} = \sqrt{b^2 - 0^2} = b.
\]

Normal component of acceleration

Note that the normal component of acceleration is equal to the magnitude of the acceleration. In other words, because the speed is constant, the acceleration is perpendicular to the velocity. See Figure 12.25.
**EXAMPLE 7  Projectile Motion**

The position vector for the projectile shown in Figure 12.26 is given by

\[ r(t) = (50\sqrt{2}t)i + (50\sqrt{2}t - 16t^2)j. \]

Find the tangential component of acceleration when \( t = 0, 1, \) and \( 25\sqrt{2}/16. \)

**Solution**

\[ v(t) = 50\sqrt{2}i + (50\sqrt{2} - 32t)j \]

\[ \|v(t)\| = 2\sqrt{50^2 - 16(50)(\sqrt{2})t + 16t^2} \]

\[ a(t) = -32j \]

The tangential component of acceleration is

\[ a_T(t) = \frac{v(t) \cdot a(t)}{\|v(t)\|} = \frac{-32(50\sqrt{2} - 32t)}{2\sqrt{50^2 - 16(50)(\sqrt{2})t + 16t^2}} \]

At the specified times, you have

\[ a_T(0) = \frac{-32(50\sqrt{2})}{100} = -16\sqrt{2} = -22.6 \]

\[ a_T(1) = \frac{-32(50\sqrt{2} - 32)}{2\sqrt{50^2 - 16(50)(\sqrt{2}) + 16}} = -15.4 \]

\[ a_T\left(\frac{25\sqrt{2}}{16}\right) = \frac{-32(50\sqrt{2} - 50\sqrt{2})}{50\sqrt{2}} = 0. \]

You can see from Figure 12.26 that, at the maximum height, when \( t = 25\sqrt{2}/16, \) the tangential component is 0. This is reasonable because the direction of motion is horizontal at the point and the tangential component of the acceleration is equal to the horizontal component of the acceleration.

**Exercises for Section 12.4**

In Exercises 1–4, sketch the unit tangent and normal vectors at the given points. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

1.  
2.  
3.  
4.  

In Exercises 5–10, find the unit tangent vector to the curve at the specified value of the parameter.

5.  \( r(t) = t^2i + 2tj, \quad t = 1 \)
6.  \( r(t) = t^2i + 2tj, \quad t = 1 \)
7.  \( r(t) = 4 \cos ti + 4 \sin tj, \quad t = \frac{\pi}{4} \)
8.  \( r(t) = 6 \cos ti + 2 \sin tj, \quad t = \frac{\pi}{3} \)
9.  \( r(t) = \ln ti + 2tj, \quad t = e \)
10.  \( r(t) = e^ti + e^tj, \quad t = 0 \)

In Exercises 11–16, find the unit tangent vector \( T(t) \) and find a set of parametric equations for the line tangent to the space curve at point \( P. \)

11.  \( r(t) = ti + t^2j + tk, \quad P(0, 0, 0) \)
12.  \( r(t) = ti + tj + \frac{1}{4}tk, \quad P(1, 1, \frac{1}{4}) \)
13.  \( r(t) = 2 \cos ti + 2 \sin tj + tk, \quad P(2, 0, 0) \)
14.  \( r(t) = \langle t, t, \sqrt{4 - t^2}, \quad P(1, 1, \sqrt{3}) \)
15.  \( r(t) = \langle 2 \cos t, 2 \sin t, 4 \rangle, \quad P(\sqrt{2}, \sqrt{2}, 4) \)
16.  \( r(t) = \langle 2 \sin t, 2 \cos t, 4 \sin^2 t, \quad P(1, \sqrt{3}, 1) \)
In Exercises 17 and 18, use a computer algebra system to graph the
space curve. Then find T(t) and find a set of parametric
equations for the line tangent to the space curve at point P.
Graph the tangent line.
17. \( \mathbf{r}(t) = (t, r^2, 2t^3/3) \), \( P(3, 9, 18) \)
18. \( \mathbf{r}(t) = 3 \cos rt + 4 \sin \mathbf{j} + \mathbf{k} \), \( P(0, 4, \pi/4) \)

Linear Approximation In Exercises 19 and 20, find a set of
parametric equations for the tangent line to the
graph at \( t = t_0 \) and use the equations for the line to approximate \( \mathbf{r}(t_0 + 0.1) \).
19. \( \mathbf{r}(t) = (t, \ln t, \sqrt{t}) \), \( t_0 = 1 \)
20. \( \mathbf{r}(t) = (e^{-t}, 2 \cos t, 2 \sin t) \), \( t_0 = 0 \)

In Exercises 21 and 22, verify that the space curves intersect at
the given values of the parameter. Find the angle between
the tangent vectors to the curves at the point of intersection.
21. \( \mathbf{r}(t) = (t^2 - 2, \frac{1}{2} t^3) \), \( t = 4 \)
\( \mathbf{u}(s) = \left( \frac{1}{2}, 2s, \sqrt{s} \right) \), \( s = 8 \)
22. \( \mathbf{r}(t) = (t, \cos t, \sin t) \), \( t = 0 \)
\( \mathbf{u}(s) = \left( -\frac{1}{2} \sin t, \cos t, \sin t \right) \), \( s = 0 \)

In Exercises 23–30, find the principal unit normal vector to the
curve at the specified value of the parameter.
23. \( \mathbf{r}(t) = t \mathbf{i} + \frac{1}{2} t^2 \mathbf{j} \), \( t = 2 \)
24. \( \mathbf{r}(t) = t \mathbf{i} + \frac{1}{4} t^2 \mathbf{j} \), \( t = 3 \)
25. \( \mathbf{r}(t) = \ln t \mathbf{i} + (t + 1) \mathbf{j} \), \( t = 2 \)
26. \( \mathbf{r}(t) = 3 \cos rt \mathbf{i} + 3 \sin t \mathbf{j} \), \( t = \frac{\pi}{4} \)
27. \( \mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + \ln r \mathbf{k} \), \( t = 1 \)
28. \( \mathbf{r}(t) = \sqrt{2} t \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{k} \), \( t = 0 \)
29. \( \mathbf{r}(t) = 6 \cos rt \mathbf{i} + 6 \sin t \mathbf{j} + \mathbf{k} \), \( t = \frac{3\pi}{4} \)
30. \( \mathbf{r}(t) = \cos t \mathbf{i} + 2 \sin t \mathbf{j} + \mathbf{k} \), \( t = -\frac{\pi}{4} \)

In Exercises 31–34, find \( v(t), a(t), \mathbf{T}(t) \), and \( \mathbf{N}(t) \) (if it exists) for
an object moving along the path given by the vector-valued function \( \mathbf{r}(t) \). Use the results to determine the form of the path.
Is the speed of the object constant or changing?
31. \( \mathbf{r}(t) = 4rt \mathbf{i} \)
32. \( \mathbf{r}(t) = 4t^2 \mathbf{i} \)
33. \( \mathbf{r}(t) = 4t^2 \mathbf{i} \)
34. \( \mathbf{r}(t) = 5t^2 \mathbf{j} + \mathbf{k} \)

In Exercises 35–44, find \( \mathbf{T}(t), \mathbf{N}(t), a_T \), and \( a_N \) at the given time
\( t \) for the plane curve \( \mathbf{r}(t) \).
35. \( \mathbf{r}(t) = t \mathbf{i} + \frac{1}{2} t^2 \mathbf{j} \), \( t = 1 \)
36. \( \mathbf{r}(t) = t^2 \mathbf{i} + 2t \mathbf{j} \), \( t = 1 \)
37. \( \mathbf{r}(t) = (t - \mathbf{i}) \mathbf{i} + 2t \mathbf{j} \), \( t = 1 \)
38. \( \mathbf{r}(t) = (t^2 - 4t) \mathbf{i} + (t^2 - 1) \mathbf{j} \), \( t = 0 \)

39. \( \mathbf{r}(t) = e^t \mathbf{i} + e^{-2t} \mathbf{j} \), \( t = 0 \)
40. \( \mathbf{r}(t) = e^t \mathbf{i} + e^{-t} \mathbf{j} + \mathbf{k} \), \( t = 0 \)
41. \( \mathbf{r}(t) = e^t \cos t \mathbf{i} + e^t \sin t \mathbf{j} \), \( t = \frac{\pi}{2} \)
42. \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + \mathbf{k} \), \( t = 0 \)
43. \( \mathbf{r}(t) = (\cos t + \sin t) \mathbf{i} + (\cos t + \sin t) \mathbf{j} \), \( t = t_0 \)
44. \( \mathbf{r}(t) = (\cos t - \sin t) \mathbf{i} + (\cos t + \sin t) \mathbf{j} \), \( t = t_0 \)

Circular Motion In Exercises 45–48, consider an object
moving according to the position function
\( \mathbf{r}(t) = \cos \omega t \mathbf{i} + \sin \omega t \mathbf{j} \).
45. Find \( \mathbf{T}(t), \mathbf{N}(t), a_T \), and \( a_N \).
46. Determine the directions of \( \mathbf{T} \) and \( \mathbf{N} \) relative to the position
function \( \mathbf{r} \).
47. Determine the speed of the object at any time \( t \) and explain its
value relative to the value of \( a_T \).
48. If the angular velocity \( \omega \) is halved, by what factor is \( a_N \)
changed?

In Exercises 49–52, sketch the graph of the plane curve given
by the vector-valued function, and, at the point on the curve
determined by \( r(t_0) \), sketch the vectors \( \mathbf{T} \) and \( \mathbf{N} \). Note that \( \mathbf{N} \)
points toward the concave side of the curve.

\begin{array}{ll}
\hline
\text{Function} & \text{Time} \\
\hline
49. \( \mathbf{r}(t) = t \mathbf{i} + \frac{1}{2} t^2 \mathbf{j} \) & \( t_0 = 2 \)
50. \( \mathbf{r}(t) = t^2 \mathbf{i} + t \mathbf{j} \) & \( t_0 = 1 \)
51. \( \mathbf{r}(t) = 2 \cos rt \mathbf{i} + 2 \sin rt \mathbf{j} \) & \( t_0 = \frac{\pi}{4} \)
52. \( \mathbf{r}(t) = 3 \cos rt \mathbf{i} + 2 \sin rt \mathbf{j} \) & \( t_0 = \pi \)
\hline
\end{array}

In Exercises 53–56, find \( \mathbf{T}(t), \mathbf{N}(t), a_T \), and \( a_N \) at the given time
\( t \) for the space curve \( \mathbf{r}(t) \). [Hint: Find \( a(t), \mathbf{T}(t), \) and \( \mathbf{N}(t) \) in the
equation \( a(t) = a_T \mathbf{T} + a_N \mathbf{N} \).]

\begin{array}{ll}
\hline
\text{Function} & \text{Time} \\
\hline
53. \( \mathbf{r}(t) = t \mathbf{i} + 2t \mathbf{j} - 3t \mathbf{k} \) & \( t = 1 \)
54. \( \mathbf{r}(t) = 4t \mathbf{i} - 4t \mathbf{j} + 2t \mathbf{k} \) & \( t = 2 \)
55. \( \mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + \frac{t^3}{2} \mathbf{k} \) & \( t = 1 \)
56. \( \mathbf{r}(t) = e^t \mathbf{i} + 2e^t \mathbf{j} + e^t \mathbf{k} \) & \( t = 0 \)
\hline
\end{array}

In Exercises 57 and 58, use a computer algebra system to graph
the space curve. Then find \( \mathbf{T}(t), \mathbf{N}(t), a_T \), and \( a_N \) at the given
time \( t \). Sketch \( \mathbf{T}(t) \) and \( \mathbf{N}(t) \) on the space curve.

\begin{array}{ll}
\hline
\text{Function} & \text{Time} \\
\hline
57. \( \mathbf{r}(t) = 4t \mathbf{i} + 3 \cos rt \mathbf{j} + 3 \sin rt \mathbf{k} \) & \( t = \frac{\pi}{2} \)
58. \( \mathbf{r}(t) = t \mathbf{i} + 3t^2 \mathbf{j} + \frac{t^3}{2} \mathbf{k} \) & \( t = 2 \)
\hline
\end{array}
Writing About Concepts
59. Define the unit tangent vector, the principal unit normal vector, and the tangential and normal components of acceleration.
60. How is the unit tangent vector related to the orientation of a curve? Explain.
61. Describe the motion of a particle if the normal component of acceleration is 0.
62. Describe the motion of a particle if the tangential component of acceleration is 0.

63. Cycloidal Motion  The figure shows the path of a particle modeled by the vector-valued function
\[ \mathbf{r}(t) = (\pi t - \sin \pi t, 1 - \cos \pi t). \]
The figure also shows the vectors \( \mathbf{v}(t)/\|\mathbf{v}(t)\| \) and \( \mathbf{a}(t)/\|\mathbf{a}(t)\| \) at the indicated values of \( t \).

(a) Find \( a_T \) and \( a_N \) at \( t = \frac{1}{2} \) and \( t = 1 \), and \( t = \frac{3}{2} \).
(b) Determine whether the speed of the particle is increasing or decreasing at each of the indicated values of \( t \). Give reasons for your answers.

64. Motion Along an Involute of a Circle  The figure shows a particle moving along a path modeled by
\[ \mathbf{r}(t) = (\cos \pi t + \pi t \sin \pi t, \sin \pi t - \pi t \cos \pi t). \]
The figure also shows the vectors \( \mathbf{v}(t) \) and \( \mathbf{a}(t) \) for \( t = 1 \) and \( t = 2 \).

(a) Find \( a_T \) and \( a_N \) at \( t = 1 \) and \( t = 2 \).
(b) Determine whether the speed of the particle is increasing or decreasing at each of the indicated values of \( t \). Give reasons for your answers.

In Exercises 65–70, find the vectors \( T \) and \( N \), and the unit binormal vector \( B = T \times N \), for the vector-valued function \( \mathbf{r}(t) \) at the given value of \( t \).

65. \( \mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + \frac{t}{2} \mathbf{k} \)
66. \( \mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + \frac{t^3}{3} \mathbf{k} \)

70. \( \mathbf{r}(t) = 2 \cos 2t \mathbf{i} + 2 \sin 2t \mathbf{j} + t \mathbf{k} \)

71. Projectile Motion  Find the tangential and normal components of acceleration for a projectile fired at an angle \( \theta \) with the horizontal at an initial speed of \( v_0 \). What are the components when the projectile is at its maximum height?

72. Projectile Motion  Use your results from Exercise 71 to find the tangential and normal components of acceleration for a projectile fired at an angle of 45° with the horizontal at an initial speed of 150 feet per second. What are the components when the projectile is at its maximum height?

73. Projectile Motion  A projectile is launched with an initial velocity of 100 feet per second at a height of 5 feet and at an angle of 30° with the horizontal.
(a) Determine the vector-valued function for the path of the projectile.
(b) Use a graphing utility to graph the path and approximate the maximum height and range of the projectile.
(c) Find \( \mathbf{v}(t), \|\mathbf{v}(t)\|, \) and \( \mathbf{a}(t) \).
(d) Use a graphing utility to complete the table.

<table>
<thead>
<tr>
<th>( t )</th>
<th>0.5</th>
<th>1.0</th>
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<th>2.0</th>
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<tr>
<td>Speed</td>
<td></td>
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</tbody>
</table>
(e) Use a graphing utility to graph the scalar functions \( a_T \) and \( a_N \). How is the speed of the projectile changing when \( a_T \) and \( a_N \) have opposite signs?
74. **Projectile Motion** A projectile is launched with an initial velocity of 200 feet per second at a height of 4 feet and at an angle of 45° with the horizontal.

(a) Determine the vector-valued function for the path of the projectile.
(b) Use a graphing utility to graph the path and approximate the maximum height and range of the projectile.
(c) Find \(v(t), \|v(t)\|,\) and \(a(t)\).
(d) Use a graphing utility to complete the table.

<table>
<thead>
<tr>
<th>(t)</th>
<th>0.5</th>
<th>1.0</th>
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<tr>
<td>Speed</td>
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</table>

75. **Air Traffic Control** Because of a storm, ground controllers instruct the pilot of a plane flying at an altitude of 4 miles to make a 90° turn and climb to an altitude of 4.2 miles. The model for the path of the plane during this maneuver is
\[
r(t) = (10 \cos \pi t, 10 \sin \pi t, 4 + 4t), \quad 0 \leq t \leq \frac{1}{\pi},
\]
where \(t\) is the time in hours and \(r\) is the distance in miles.

(a) Determine the speed of the plane.
(b) Use a computer algebra system to calculate \(a_T\) and \(a_N\). Why is one of these equal to 0?

76. **Projectile Motion** A plane flying at an altitude of 36,000 feet at a speed of 600 miles per hour releases a bomb. Find the tangential and normal components of acceleration acting on the bomb.

77. **Centripetal Acceleration** An object is spinning at a constant speed on the end of a string, according to the position function given in Exercises 45–48.

(a) If the angular velocity \(\omega\) is doubled, how is the centripetal component of acceleration changed?
(b) If the angular velocity is unchanged but the length of the string is halved, how is the centripetal component of acceleration changed?

78. **Centripetal Force** An object of mass \(m\) moves at a constant speed \(v\) in a circular path of radius \(r\). The force required to produce the centripetal component of acceleration is called the centripetal force and is given by \(F = \frac{mv^2}{r}\). Newton’s Law of Universal Gravitation is given by \(F = \frac{GMm}{d^2}\), where \(d\) is the distance between the centers of the two bodies of masses \(M\) and \(m\), and \(G\) is a gravitational constant. Use this law to show that the speed required for circular motion is \(v = \sqrt{\frac{GM}{r}}\).

**Orbital Speed** In Exercises 79–82, use the result of Exercise 78 to find the speed necessary for the given circular orbit around Earth. Let \(GM = 9.56 \times 10^5\) cubic miles per second per second, and assume the radius of Earth is 4000 miles.

79. The orbit of a space shuttle 100 miles above the surface of Earth
80. The orbit of a space shuttle 200 miles above the surface of Earth
81. The orbit of a heat capacity mapping satellite 385 miles above the surface of Earth
82. The orbit of a SYNCOM satellite \(r\) miles above the surface of Earth that is in geosynchronous orbit (The satellite completes one orbit per sidereal day (approximately 23 hours, 56 minutes), and therefore appears to remain stationary above a point on Earth.)

**True or False?** In Exercises 83 and 84, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

83. If a car’s speedometer is constant, then the car cannot be accelerating.
84. If \(a_N = 0\) for a moving object, then the object is moving in a straight line.

85. A particle moves along a path modeled by
\[
r(t) = \cosh(bt)i + \sinh(bt)j
\]
where \(b\) is a positive constant.

(a) Show that the path of the particle is a hyperbola.
(b) Show that \(a(t) = b^2 r(t)\).

86. Prove that the principal unit normal vector \(N\) points toward the concave side of a plane curve.
87. Prove that the vector \(T(t)\) is \(0\) for an object moving in a straight line.
88. Prove that \(a_N = \frac{\|v \times a\|}{\|v\|} \cdot a_T\).
89. Prove that \(a_N = \sqrt{\|a\|^2 - a_T^2}\).

**Putnam Exam Challenge**

90. A particle of unit mass moves on a straight line under the action of a force which is a function \(f(v)\) of the velocity \(v\) of the particle, but the form of this function is not known. A motion is observed, and the distance \(x\) covered in time \(t\) is found to be connected with \(t\) by the formula \(x = at + bt^2 + ct^3\), where \(a, b,\) and \(c\) have numerical values determined by observation of the motion. Find the function \(f(v)\) for the range of \(v\) covered by the experiment.

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