Arc Length and Curvature

- Find the arc length of a space curve.
- Use the arc length parameter to describe a plane curve or space curve.
- Find the curvature of a curve at a point on the curve.
- Use a vector-valued function to find frictional force.

Arc Length

In Section 10.3, you saw that the arc length of a smooth plane curve \( C \) given by the parametric equations \( x = x(t) \) and \( y = y(t) \), \( a \leq t \leq b \), is

\[
s = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.
\]

In vector form, where \( C \) is given by \( \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \), you can rewrite this equation for arc length as

\[
s = \int_{a}^{b} \| \mathbf{r}'(t) \| \, dt.
\]

The formula for the arc length of a plane curve has a natural extension to a smooth curve in space, as stated in the following theorem.

**Theorem 12.6** Arc Length of a Space Curve

If \( C \) is a smooth curve given by \( \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \), on an interval \([a, b]\), then the arc length of \( C \) on the interval is

\[
s = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt = \int_{a}^{b} \| \mathbf{r}'(t) \| \, dt.
\]

**Example 1** Finding the Arc Length of a Curve in Space

Find the arc length of the curve given by

\[
\mathbf{r}(t) = t\mathbf{i} + \frac{4}{3} t^{3/2}\mathbf{j} + \frac{1}{2} t^2 \mathbf{k}
\]

from \( t = 0 \) to \( t = 2 \), as shown in Figure 12.27.

**Solution** Using \( x(t) = t \), \( y(t) = \frac{4}{3} t^{3/2} \), and \( z(t) = \frac{1}{2} t^2 \), you obtain \( x'(t) = 1 \), \( y'(t) = 2t^{1/2} \), and \( z'(t) = t \). So, the arc length from \( t = 0 \) to \( t = 2 \) is given by

\[
s = \int_{0}^{2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt \quad \text{Formula for arc length}
\]

\[
= \int_{0}^{2} \sqrt{1 + 4t + t^2} \, dt
\]

\[
= \int_{0}^{2} \sqrt{(t + 2)^2 - 3} \, dt
\]

\[
= \left[ \frac{1 + 2}{2} \sqrt{(t + 2)^2 - 3} - \frac{3}{2} \ln(t + 2) + \sqrt{(t + 2)^2 - 3} \right]_{0}^{2}
\]

\[
= 2\sqrt{3} - \frac{3}{2} \ln(4 + \sqrt{13}) - 1 + \frac{3}{2} \ln 3 \approx 4.816.
\]
EXAMPLE 2  Finding the Arc Length of a Helix

Find the length of one turn of the helix given by
\[ \mathbf{r}(t) = b \cos t \mathbf{i} + b \sin t \mathbf{j} + \sqrt{1 - b^2} t \mathbf{k} \]
as shown in Figure 12.28.

Solution  Begin by finding the derivative.
\[ \mathbf{r}'(t) = -b \sin t \mathbf{i} + b \cos t \mathbf{j} + \frac{\sqrt{1 - b^2}}{2} \]

Now, using the formula for arc length, you can find the length of one turn of the helix by integrating \( \| \mathbf{r}'(t) \| \) from 0 to 2\( \pi \).

\[
\begin{align*}
\mathbf{s} &= \int_0^{2\pi} \| \mathbf{r}'(t) \| \, dt \\
&= \int_0^{2\pi} \sqrt{b^2 \sin^2 t + \cos^2 t} \, dt + (1 - b^2) \, dt \\
&= \int_0^{2\pi} \, dt \\
&= 2\pi.
\end{align*}
\]

So, the length is 2\( \pi \) units.

Arc Length Parameter

You have seen that curves can be represented by vector-valued functions in different ways, depending on the choice of parameter. For motion along a curve, the convenient parameter is time \( t \). However, for studying the geometric properties of a curve, the convenient parameter is often arc length \( s \).

Definition of Arc Length Function

Let \( C \) be a smooth curve given by \( \mathbf{r}(t) \) defined on the closed interval \( [a, b] \). For \( a \leq t \leq b \), the arc length function is given by
\[
\mathbf{s}(t) = \int_a^t \| \mathbf{r}'(u) \| \, du = \int_a^t \sqrt{\left( \frac{dx}{du}(u) \right)^2 + \left( \frac{dy}{du}(u) \right)^2 + \left( \frac{dz}{du}(u) \right)^2} \, du.
\]

The arc length \( s \) is called the arc length parameter. (See Figure 12.29.)

NOTE  The arc length function \( s \) is nonnegative. It measures the distance along \( C \) from the initial point \((x(a), y(a), z(a))\) to the point \((x(t), y(t), z(t))\).

Using the definition of the arc length function and the Second Fundamental Theorem of Calculus, you can conclude that
\[
\frac{ds}{dt} = \| \mathbf{r}'(t) \|.
\]

In differential form, you can write
\[
ds = \| \mathbf{r}'(t) \| \, dt.
\]
EXAMPLE 3  Finding the Arc Length Function for a Line

Find the arc length function $s(t)$ for the line segment given by

$$\mathbf{r}(t) = (3 - 3t)i + 4tj, \quad 0 \leq t \leq 1$$

and write $\mathbf{r}$ as a function of the parameter $s$. (See Figure 12.30.)

Solution  Because $\mathbf{r}'(t) = -3i + 4j$ and

$$||\mathbf{r}'(t)|| = \sqrt{(-3)^2 + 4^2} = 5$$

you have

$$s(t) = \int_0^t ||\mathbf{r}'(u)|| \, du = \int_0^t 5 \, du = 5t.$$  

Using $s = 5t$ (or $t = s/5$), you can rewrite $\mathbf{r}$ using the arc length parameter as follows.

$$\mathbf{r}(s) = \left(3 - \frac{3}{5}s\right)i + \frac{4}{5}sj, \quad 0 \leq s \leq 5.$$  

One of the advantages of writing a vector-valued function in terms of the arc length parameter is that $||\mathbf{r}'(s)|| = 1$. For instance, in Example 3, you have

$$||\mathbf{r}'(s)|| = \sqrt{\left(-\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1.$$  

So, for a smooth curve $C$ represented by $\mathbf{r}(s)$, where $s$ is the arc length parameter, the arc length between $a$ and $b$ is

$$\text{Length of arc} = \int_a^b ||\mathbf{r}'(s)|| \, ds = \int_a^b ds = b - a,$$

where $b - a$ is the length of interval.

Furthermore, if $t$ is any parameter such that $||\mathbf{r}'(t)|| = 1$, then $t$ must be the arc length parameter. These results are summarized in the following theorem, which is stated without proof.

\textbf{THEOREM 12.7  Arc Length Parameter}

If $C$ is a smooth curve given by

$$\mathbf{r}(s) = x(s)i + y(s)j \quad \text{or} \quad \mathbf{r}(s) = x(s)i + y(s)j + z(s)k$$

where $s$ is the arc length parameter, then

$$||\mathbf{r}'(s)|| = 1.$$  

Moreover, if $t$ is any parameter for the vector-valued function $\mathbf{r}$ such that $||\mathbf{r}'(t)|| = 1$, then $t$ must be the arc length parameter.
Curvature

An important use of the arc length parameter is to find curvature—the measure of how sharply a curve bends. For instance, in Figure 12.31 the curve bends more sharply at \( P \) than at \( Q \), and you can say that the curvature is greater at \( P \) than at \( Q \). You can calculate curvature by calculating the magnitude of the rate of change of the unit tangent vector with respect to the arc length as shown in Figure 12.32.

A circle has the same curvature at any point. Moreover, the curvature and the radius of the circle are inversely related. That is, a circle with a large radius has a small curvature, and a circle with a small radius has a large curvature. This inverse relationship is made explicit in the following example.

**EXAMPLE 4 Finding the Curvature of a Circle**

Show that the curvature of a circle of radius \( r \) is \( K = 1/r \).

**Solution** Without loss of generality you can consider the circle to be centered at the origin. Let \((x, y)\) be any point on the circle and let \( s \) be the length of the arc from \((r, 0)\) to \((x, y)\), as shown in Figure 12.33. By letting \( \theta \) be the central angle of the circle, you can represent the circle by

\[
r(\theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j}, \quad \theta \text{ is the parameter.}
\]

Using the formula for the length of a circular arc \( s = r \theta \), you can rewrite \( r(\theta) \) in terms of the arc length parameter as follows.

\[
r(s) = r \cos \frac{s}{r} \mathbf{i} + r \sin \frac{s}{r} \mathbf{j}, \quad \text{Arc length } s \text{ is the parameter.}
\]

So, \( r'(s) = -\sin \frac{s}{r} \mathbf{i} + \cos \frac{s}{r} \mathbf{j} \), and it follows that \( ||r'(s)|| = 1 \), which implies that the unit tangent vector is

\[
T(s) = \frac{r'(s)}{||r'(s)||} = -\sin \frac{s}{r} \mathbf{i} + \cos \frac{s}{r} \mathbf{j}
\]

and the curvature is given by

\[
K = ||T'(s)|| = \left| -\frac{1}{r} \cos \frac{s}{r} \mathbf{i} - \frac{1}{r} \sin \frac{s}{r} \mathbf{j} \right| = \frac{1}{r}
\]
at every point on the circle.

**NOTE** Because a straight line doesn’t curve, you would expect its curvature to be 0. Try checking this by finding the curvature of the line given by

\[
r(s) = \left(3 - \frac{3}{5}s\right)\mathbf{i} + \frac{4}{5}s\mathbf{j}.
\]
In Example 4, the curvature was found by applying the definition directly. This requires the curve be written in terms of the arc length parameter s. The following theorem gives two other formulas for finding the curvature of a curve written in terms of an arbitrary parameter t. The proof of this theorem is left as an exercise [see Exercise 88, parts (a) and (b)].

**THEOREM 12.8 Formulas for Curvature**

If C is a smooth curve given by \( \mathbf{r}(t) \), then the curvature \( K \) of C at t is given by

\[
K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.
\]

Because \( \|\mathbf{r}'(t)\| = ds/dt \), the first formula implies that curvature is the ratio of the rate of change in the tangent vector \( \mathbf{T} \) to the rate of change in arc length. To see that this is reasonable, let \( \Delta t \) be a “small number.” Then,

\[
\frac{\mathbf{T}(t)}{ds/dt} = \frac{\mathbf{T}(t + \Delta t) - \mathbf{T}(t)}{s(t + \Delta t) - s(t)}/\Delta t = \frac{\mathbf{T}(t + \Delta t) - \mathbf{T}(t)}{s(t + \Delta t) - s(t)} = \frac{\Delta \mathbf{T}}{\Delta s}.
\]

In other words, for a given \( \Delta s \), the greater the length of \( \Delta \mathbf{T} \), the more the curve bends at \( t \), as shown in Figure 12.34.

**EXAMPLE 5 Finding the Curvature of a Space Curve**

Find the curvature of the curve given by \( \mathbf{r}(t) = 2t \mathbf{i} + t^2 \mathbf{j} - \frac{1}{2} t^3 \mathbf{k} \).

**Solution**  It is not apparent whether this parameter represents arc length, so you should use the formula \( K = \|\mathbf{T}'(t)\|/\|\mathbf{r}'(t)\| \).

\[
\mathbf{r}'(t) = 2 \mathbf{i} + 2t \mathbf{j} - t^2 \mathbf{k}
\]

\[
\|\mathbf{r}'(t)\| = \sqrt{4 + 4t^2 + t^4} = t^2 + 2 \\
\text{Length of } \mathbf{r}'(t)
\]

\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{2 \mathbf{i} + 2t \mathbf{j} - t^2 \mathbf{k}}{t^2 + 2}
\]

\[
\mathbf{T}'(t) = \frac{(t^2 + 2)(2 \mathbf{j} - 2t \mathbf{k}) - (2t)(2 \mathbf{i} + 2t \mathbf{j} - t^2 \mathbf{k})}{(t^2 + 2)^2}
\]

\[
= \frac{-4t \mathbf{i} + (4 - 2t^2) \mathbf{j} - 4t \mathbf{k}}{(t^2 + 2)^2}
\]

\[
\|\mathbf{T}'(t)\| = \frac{\sqrt{16t^2 + 16 - 16t^2 + 4t^4 + 16t^2}}{(t^2 + 2)^2}
\]

\[
= \frac{2(t^2 + 2)}{(t^2 + 2)^2}
\]

\[
= \frac{2}{t^2 + 2} \\
\text{Length of } \mathbf{T}'(t)
\]

Therefore,

\[
K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{2}{(t^2 + 2)^2}.
\]

Curvature
The following theorem presents a formula for calculating the curvature of a plane curve given by \( y = f(x) \).

**THEOREM 12.9 Curvature in Rectangular Coordinates**

If \( C \) is the graph of a twice-differentiable function given by \( y = f(x) \), then the curvature \( K \) at the point \((x, y)\) is given by

\[
K = \frac{|y''|}{(1 + (y')^2)^{3/2}}.
\]

**Proof** By representing the curve \( C \) by \( r(x) = x\mathbf{i} + f(x)\mathbf{j} + 0\mathbf{k} \) (where \( x \) is the parameter), you obtain \( r'(x) = \mathbf{i} + f'(x)\mathbf{j} \),

\[
\|r'(x)\| = \sqrt{1 + [f'(x)]^2}
\]

and \( r''(x) = f''(x)\mathbf{j} \). Because \( r'(x) \times r''(x) = f''(x)\mathbf{k} \), it follows that the curvature is

\[
K = \frac{\|r'(x) \times r''(x)\|}{\|r'(x)\|^3} = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}.
\]

Let \( C \) be a curve with curvature \( K \) at point \( P \). The circle passing through point \( P \) with radius \( r = 1/K \) is called the **circle of curvature** if the circle lies on the concave side of the curve and shares a common tangent line with the curve at point \( P \). The radius is called the **radius of curvature** at \( P \), and the center of the circle is called the **center of curvature**.

The circle of curvature gives you a nice way to estimate graphically the curvature \( K \) at a point \( P \) on a curve. Using a compass, you can sketch a circle that lies against the concave side of the curve at point \( P \), as shown in Figure 12.35. If the circle has a radius of \( r \), you can estimate the curvature to be \( K = 1/r \).

**EXAMPLE 6 Finding Curvature in Rectangular Coordinates**

Find the curvature of the parabola given by \( y = -\frac{1}{3}x^2 \) at \( x = 2 \). Sketch the circle of curvature at \((2, 1)\).

**Solution** The curvature at \( x = 2 \) is as follows.

\[
y' = 1 - \frac{x}{2} \quad y' = 0
\]

\[
y'' = -\frac{1}{2} \quad y'' = -\frac{1}{2}
\]

\[
K = \frac{|y''|}{(1 + (y')^2)^{3/2}} = \frac{1}{2}
\]

Because the curvature at \( P(2, 1) \) is \( \frac{1}{2} \), it follows that the radius of the circle of curvature at that point is 2. So, the center of curvature is \((2, -1)\), as shown in Figure 12.36. [In the figure, note that the curve has the greatest curvature at \( P \). Try showing that the curvature at \( Q(4, 0) \) is \( 1/2^{5/2} \approx 0.177 \).]
Arc length and curvature are closely related to the tangential and normal components of acceleration. The tangential component of acceleration is the rate of change of the speed, which in turn is the rate of change of the arc length. This component is negative as a moving object slows down and positive as it speeds up—regardless of whether the object is turning or traveling in a straight line. So, the tangential component is solely a function of the arc length and is independent of the curvature.

On the other hand, the normal component of acceleration is a function of both speed and curvature. This component measures the acceleration acting perpendicular to the direction of motion. To see why the normal component is affected by both speed and curvature, imagine that you are driving a car around a turn, as shown in Figure 12.37. If your speed is high and the turn is sharp, you feel yourself thrown against the car door. By lowering your speed taking a more gentle turn, you are able to lessen this sideways thrust.

The next theorem explicitly states the relationships among speed, curvature, and the components of acceleration.

**Theorem 12.10 Acceleration, Speed, and Curvature**

If \( \mathbf{r}(t) \) is the position vector for a smooth curve \( C \), then the acceleration vector is given by

\[
\mathbf{a}(t) = \frac{d^2s}{dt^2} \mathbf{T} + K \left( \frac{ds}{dt} \right)^2 \mathbf{N}
\]

where \( K \) is the curvature of \( C \) and \( ds/dt \) is the speed.

**Proof** For the position vector \( \mathbf{r}(t) \), you have

\[
\mathbf{a}(t) = a_T \mathbf{T} + a_N \mathbf{N} = D_s(\mathbf{v} \cdot \mathbf{T}) + \|\mathbf{v}\| \mathbf{T} \cdot \mathbf{N} = \frac{d^2s}{dt^2} \mathbf{T} + \frac{ds}{dt} \|\mathbf{v}\|K \mathbf{N} = \frac{d^2s}{dt^2} \mathbf{T} + K \left( \frac{ds}{dt} \right)^2 \mathbf{N}.
\]

**Example 7 Tangential and Normal Components of Acceleration**

Find \( a_T \) and \( a_N \) for the curve given by

\[
\mathbf{r}(t) = 2t \mathbf{i} + t^2 \mathbf{j} - \frac{1}{2} t^3 \mathbf{k}.
\]

**Solution** From Example 5, you know that

\[
\frac{ds}{dt} = \|\mathbf{r}'(t)\| = t^2 + 2 \quad \text{and} \quad K = \frac{2}{(t^2 + 2)^2}.
\]

Therefore,

\[
a_T = \frac{d^2s}{dt^2} = 2t \quad \text{Tangential component}
\]

and

\[
a_N = K \left( \frac{ds}{dt} \right)^2 = \frac{2}{(t^2 + 2)^2} (t^2 + 2)^2 = 2. \quad \text{Normal component}
\]
Application

There are many applications in physics and engineering dynamics that involve the relationships among speed, arc length, curvature, and acceleration. One such application concerns frictional force.

A moving object with mass \( m \) is in contact with a stationary object. The total force required to produce an acceleration \( \mathbf{a} \) along a given path is

\[
F = ma = m \left( \frac{d^2s}{dt^2} \right) T + mK \left( \frac{ds}{dt} \right)^2 N
\]

\[
= ma_T T + ma_N N.
\]

The portion of this total force that is supplied by the stationary object is called the force of friction. For example, if a car moving with constant speed is rounding a turn, the roadway exerts a frictional force that keeps the car from sliding off the road. If the car is not sliding, the frictional force is perpendicular to the direction of motion and has magnitude equal to the normal component of acceleration, as shown in Figure 12.38. The potential frictional force of a road around a turn can be increased by banking the roadway.

**EXAMPLE 8 Frictional Force**

A 360-kilogram go-cart is driven at a speed of 60 kilometers per hour around a circular racetrack of radius 12 meters, as shown in Figure 12.39. To keep the cart from skidding off course, what frictional force must the track surface exert on the tires?

**Solution** The frictional force must equal the mass times the normal component of acceleration. For this circular path, you know that the curvature is

\[
K = \frac{1}{12}.
\]

Curvature of circular racetrack

Therefore, the frictional force is

\[
ma_N = mK \left( \frac{ds}{dt} \right)^2
\]

\[
= (360 \text{ kg}) \left( \frac{1}{12 \text{ m}} \right) \left( \frac{60,000 \text{ m}}{3600 \text{ sec}} \right)^2
\]

\[
\approx 8333 \text{ (kg)(m)/sec}^2.
\]
### Summary of Velocity, Acceleration, and Curvature

Let $C$ be a curve (in the plane or in space) given by the position function
\[
\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \quad \text{Curve in the plane}
\]
\[
\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}. \quad \text{Curve in space}
\]

**Velocity vector, speed, and acceleration vector:**

\[
\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} \quad \text{Velocity vector}
\]
\[
\|\mathbf{v}(t)\| = \frac{ds}{dt} \quad \text{Speed}
\]
\[
\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} + a_N \mathbf{N}(t) \quad \text{Acceleration vector}
\]

**Unit tangent vector and principal unit normal vector:**

\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \quad \text{and} \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}
\]

**Components of acceleration:**

\[
a_T = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{d^2s}{dt^2}
\]
\[
a_N = \mathbf{a} \cdot \mathbf{N} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_T^2} = K\left(\frac{ds}{dt}\right)^2
\]

**Formulas for curvature in the plane:**

\[
K = \frac{|\mathbf{r}'\mathbf{r}'' - (\mathbf{r}'\mathbf{r}')^2|}{\|\mathbf{r}'\|^3} \quad C \text{ given by } y = f(x)
\]
\[
K = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}} \quad C \text{ given by } x = x(t), y = y(t)
\]

**Formulas for curvature in the plane or in space:**

\[
K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \quad s \text{ is arc length parameter.}
\]
\[
K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \quad \tau \text{ is general parameter.}
\]
\[
K = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}(t)\|^2}
\]

Cross product formulas apply only to curves in space.

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### Exercises for Section 12.5

**In Exercises 1–6, sketch the plane curve and find its length over the given interval.**

<table>
<thead>
<tr>
<th>Function</th>
<th>Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j}$</td>
<td>$[0, 4]$</td>
</tr>
<tr>
<td>2. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$</td>
<td>$[0, 4]$</td>
</tr>
<tr>
<td>3. $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$</td>
<td>$[0, 2]$</td>
</tr>
<tr>
<td>4. $\mathbf{r}(t) = (t + 1)\mathbf{i} + t^2\mathbf{j}$</td>
<td>$[0, 6]$</td>
</tr>
<tr>
<td>5. $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j}$</td>
<td>$[0, 2\pi]$</td>
</tr>
<tr>
<td>6. $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$</td>
<td>$[0, 2\pi]$</td>
</tr>
</tbody>
</table>

**7. Projectile Motion** A baseball is hit 3 feet above the ground at 100 feet per second and at an angle of $45^\circ$ with respect to the ground.

(a) Find the vector-valued function for the path of the baseball.
(b) Find the maximum height.
(c) Find the range.
(d) Find the arc length of the trajectory.

**8. Projectile Motion** An object is launched from ground level. Determine the angle of the launch to obtain (a) the maximum height, (b) the maximum range, and (c) the maximum length of the trajectory. For part (c), let $v_0 = 96$ feet per second.

**In Exercises 9–14, sketch the space curve and find its length over the given interval.**

<table>
<thead>
<tr>
<th>Function</th>
<th>Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>9. $\mathbf{r}(t) = 2\mathbf{i} - 3\mathbf{j} + t\mathbf{k}$</td>
<td>$[0, 2]$</td>
</tr>
<tr>
<td>10. $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + t^3\mathbf{k}$</td>
<td>$[0, 2]$</td>
</tr>
<tr>
<td>11. $\mathbf{r}(t) = (3t, 2\cos t, 2\sin t)$</td>
<td>$[0, \frac{\pi}{2}]$</td>
</tr>
<tr>
<td>12. $\mathbf{r}(t) = (2\sin t, 5t, 2\cos t)$</td>
<td>$[0, \pi]$</td>
</tr>
<tr>
<td>13. $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}$</td>
<td>$[0, 2\pi]$</td>
</tr>
<tr>
<td>14. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t^2 \rangle$</td>
<td>$[0, \frac{\pi}{2}]$</td>
</tr>
</tbody>
</table>
In Exercises 15 and 16, use the integration capabilities of a graphing utility to approximate the length of the space curve over the given interval.

<table>
<thead>
<tr>
<th>Function</th>
<th>Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{r}(t) = t^2 \mathbf{i} + tj + \ln rk )</td>
<td>( 1 \leq t \leq 3 )</td>
</tr>
<tr>
<td>( \mathbf{r}(t) = \sin 3t \mathbf{i} + \cos 3t \mathbf{j} + t^3 \mathbf{k} )</td>
<td>( 0 \leq t \leq 2 )</td>
</tr>
</tbody>
</table>

17. **Investigation** Consider the graph of the vector-valued function

\[ \mathbf{r}(t) = ti + (4-t^2)j + t^4k \]

on the interval \([0, 2]\).

(a) Approximate the length of the curve by finding the length of the line segment connecting its endpoints.

(b) Approximate the length of the curve by summing the lengths of the line segments connecting the terminal points of the vectors \( \mathbf{r}(0) \), \( \mathbf{r}(0.5) \), \( \mathbf{r}(1) \), \( \mathbf{r}(1.5) \), and \( \mathbf{r}(2) \).

(c) Describe how you could obtain a more accurate approximation by continuing the processes in parts (a) and (b).

(d) Use the integration capabilities of a graphing utility to approximate the length of the curve. Compare this result with the answers in parts (a) and (b).

18. **Investigation** Repeat Exercise 17 for the vector-valued function \( \mathbf{r}(t) = 6 \cos (\pi t/4) \mathbf{i} + 2 \sin (\pi t/4) \mathbf{j} + rk \).

19. **Investigation** Consider the helix represented by the vector-valued function \( \mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle \).

(a) Write the length of the arc \( s \) on the helix as a function of \( t \) by evaluating the integral

\[ s = \int_0^t \sqrt{\langle x(u) \rangle^2 + \langle y(u) \rangle^2 + \langle z(u) \rangle^2} \, du. \]

(b) Solve for \( t \) in the relationship derived in part (a), and substitute the result into the original set of parametric equations. This yields a parametrization of the curve in terms of the arc length parameter \( s \).

(c) Find the coordinates of the point on the helix for arc lengths \( s = \sqrt{2} \) and \( s = 4 \).

(d) Verify that \( \| \mathbf{r}'(s) \| = 1 \).

20. **Investigation** Repeat Exercise 19 for the curve represented by the vector-valued function

\[ \mathbf{r}(t) = \langle 4 \sin t - t \cos t, 4 \cos t + t \sin t \rangle, \frac{1}{2} t^2 \rangle. \]

In Exercises 21–24, find the curvature \( K \) of the curve, where \( s \) is the arc length parameter.

21. \( \mathbf{r}(s) = \left( 1 + \frac{\sqrt{3}}{2} s \right) \mathbf{i} + \left( 1 - \frac{\sqrt{3}}{2} \right) \mathbf{j} \)

22. \( \mathbf{r}(s) = (3 + s) \mathbf{i} + \mathbf{j} \)

23. Helix in Exercise 19: \( \mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle \)

24. Curve in Exercise 20:

\[ \mathbf{r}(t) = \langle 4 \sin t - t \cos t, 4 \cos t + t \sin t, \frac{1}{2} t^2 \rangle \]

In Exercises 25–30, find the curvature \( K \) of the plane curve at the given value of the parameter.

25. \( \mathbf{r}(t) = 4t \mathbf{i} - 2t \mathbf{j}, \quad t = 1 \)

26. \( \mathbf{r}(t) = t^3 \mathbf{j} + \mathbf{k}, \quad t = 0 \)

27. \( \mathbf{r}(t) = ti + \frac{1}{t} \mathbf{j}, \quad t = 1 \)

28. \( \mathbf{r}(t) = ti + t^2 \mathbf{j}, \quad t = 1 \)

29. \( \mathbf{r}(t) = ti + \cos tj, \quad t = 0 \)

30. \( \mathbf{r}(t) = 5 \cos ti + 4 \sin tj, \quad t = \frac{\pi}{3} \)

In Exercises 31–40, find the curvature \( K \) of the curve.

31. \( \mathbf{r}(t) = 4 \cos 2\pi t \mathbf{i} + 4 \sin 2\pi tj \)

32. \( \mathbf{r}(t) = 2 \cos \pi ti + \sin \pi tj \)

33. \( \mathbf{r}(t) = a \cos at i + a \sin at \mathbf{j} \)

34. \( \mathbf{r}(t) = a \cos at i + b \sin at \mathbf{j} \)

35. \( \mathbf{r}(t) = \langle a \cos at - \sin at, a(1 - \cos at) \rangle \)

36. \( \mathbf{r}(t) = \langle \cos at + at \sin at, \sin at - at \cos at \rangle \)

37. \( \mathbf{r}(t) = ti + t^2 \mathbf{j} + \frac{t^3}{2} \mathbf{k} \)

38. \( \mathbf{r}(t) = 2t^2 \mathbf{i} + tj + \frac{t^3}{2} \mathbf{k} \)

39. \( \mathbf{r}(t) = 4ti + 3 \cos tj + 3 \sin rk \)

40. \( \mathbf{r}(t) = e^t \cos ti + e^t \sin tj + e^t \mathbf{k} \)

In Exercises 41–46, find the curvature and radius of curvature of the plane curve at the given value of \( x \).

41. \( y = 3x - 2, \quad x = a \)

42. \( y = mx + b, \quad x = a \)

43. \( y = 2x^2 + 3, \quad x = -1 \)

44. \( y = 2x + \frac{4}{x}, \quad x = 1 \)

45. \( y = \sqrt{a^2 - x^2}, \quad x = 0 \)

46. \( y = \frac{1}{2} \sqrt{16 - x^2}, \quad x = 0 \)

**Writing** In Exercises 47 and 48, two circles of curvature to the graph of the function are given. (a) Find the equation of the smaller circle, and (b) write a short paragraph explaining why the circles have different radii.

47. \( f(x) = \sin x \)

48. \( f(x) = 4x^2/(x^2 + 3) \)
In Exercises 49–52, use a graphing utility to graph the function. In the same viewing window, graph the circle of curvature to the graph at the given value of $x$.

49. $y = x + \frac{1}{x}, \quad x = 1$

50. $y = \ln x, \quad x = 1$

51. $y = e^t, \quad x = 0$

52. $y = \frac{1}{2}x^3, \quad x = 1$

**Evolute** An evolute is the curve formed by the set of centers of curvature of a curve. In Exercises 53 and 54, a curve and its evolute are given. Use a compass to sketch the circles of curvature at points $A$ and $B$. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

53. Cycloid: $x = t - \sin t$
   $y = 1 - \cos t$

   Evolute: $x = \sin t + t$
   $y = \cos t - 1$

54. Ellipse: $x = 3\cos t$
   $y = 2\sin t$

   Evolute: $x = \frac{3}{4}\cos^3 t$
   $y = \frac{5}{2}\sin^3 t$

In Exercises 55–60, (a) find the point on the curve at which the curvature $K$ is a maximum and (b) find the limit of $K$ as $x \to \infty$.

55. $y = (x - 1)^2 + 3$

56. $y = x^3$

57. $y = x^{2/3}$

58. $y = \frac{1}{x}$

59. $y = \ln x$

60. $y = e^x$

In Exercises 61–64, find all points on the graph of the function such that the curvature is zero.

61. $y = 1 - x^3$

62. $y = (x - 1)^3 + 3$

63. $y = \cos x$

64. $y = \sin x$

**Writing About Concepts**

65. Describe the graph of a vector-valued function for which the curvature is 0 for all values of $t$ in its domain.

66. Given a twice-differentiable function $y = f(x)$, determine its curvature at a relative extremum. Can the curvature ever be greater than it is at a relative extremum? Why or why not?
88. Use the definition of curvature in space, \( K = \|\mathbf{T}'(s)\|/\|\mathbf{r}'(s)\| \), to verify each formula.
(a) \( K = \|\mathbf{T}'(t)\|/\|\mathbf{r}'(t)\| \)
(b) \( K = \|\mathbf{r}'(t) \times \mathbf{r}''(t)\|/\|\mathbf{r}'(t)\|^3 \)
(c) \( K = \mathbf{a}(t) \cdot \mathbf{N}(t)/\|\mathbf{v}(t)\|^3 \)

**True or False?** In Exercises 89–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

89. The arc length of a space curve depends on the parametrization.
90. The curvature of a circle is the same as its radius.
91. The curvature of a line is 0.
92. The normal component of acceleration is a function of both speed and curvature.

**Kepler’s Laws** In Exercises 93–100, you are asked to verify Kepler’s Laws of Planetary Motion. For these exercises, assume that each planet moves in an orbit given by the vector-valued function \( \mathbf{r} \). Let \( r = \|\mathbf{r}\| \), let \( G \) represent the universal gravitational constant, let \( M \) represent the mass of the sun, and let \( m \) represent the mass of the planet.

93. Prove that \( \mathbf{r} \cdot \mathbf{r}' = \frac{dr}{dt} \)
94. Using Newton’s Second Law of Motion, \( \mathbf{F} = m\mathbf{a} \), and Newton’s Second Law of Gravitation, \( \mathbf{F} = -(GmM/r^2)\mathbf{r} \), show that \( \mathbf{a} \) and \( \mathbf{r} \) are parallel, and that \( \mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{L} \) is a constant vector. So, \( \mathbf{r}'(t) \) moves in a fixed plane, orthogonal to \( \mathbf{L} \).
95. Prove that \( \frac{d}{dt} \left( \frac{\mathbf{r}}{r} \right) = \frac{1}{r^3} \left( [\mathbf{r} \times \mathbf{r}'] \times \mathbf{r} \right) \).
96. Show that \( \frac{\mathbf{r}'}{GM} = \mathbf{L} - \frac{\mathbf{r}}{r} = \mathbf{e} \) is a constant vector.
97. Prove Kepler’s First Law: Each planet moves in an elliptical orbit with the sun as a focus.
98. Assume that the elliptical orbit
\[
 r = \frac{ed}{1 + e \cos \theta}
\]
is in the xy-plane, with \( \mathbf{L} \) along the z-axis. Prove that
\[
\|\mathbf{L}\| = r^2 \frac{d\theta}{dt}
\]
99. Prove Kepler’s Second Law: Each ray from the sun to a planet sweeps out equal areas of the ellipse in equal times.
100. Prove Kepler’s Third Law: The square of the period of a planet’s orbit is proportional to the cube of the mean distance between the planet and the sun.