Section 3.5

Limits at Infinity

- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

Limits at Infinity

This section discusses the “end behavior” of a function on an infinite interval. Consider the graph of
\[ f(x) = \frac{3x^2}{x^2 + 1} \]
as shown in Figure 3.33. Graphically, you can see that the values of \( f(x) \) appear to approach 3 as \( x \) increases without bound or decreases without bound. You can come to the same conclusions numerically, as shown in the table.

The table suggests that the value of \( f(x) \) approaches 3 as \( x \) increases without bound and as \( x \) decreases without bound.

The statement \( \lim_{x \to \infty} f(x) = L \)
or \( \lim_{x \to \infty} f(x) = L \), means that the limit exists and the limit is equal to \( L \).

NOTE The statement \( \lim_{x \to \infty} f(x) = L \)

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The table suggests that the value of \( f(x) \) approaches 3 as \( x \) increases without bound and as \( x \) decreases without bound.

These limits at infinity are denoted by

\[ \lim_{x \to \infty} f(x) = 3 \quad \text{Limit at negative infinity} \]

and

\[ \lim_{x \to -\infty} f(x) = 3. \quad \text{Limit at positive infinity} \]

To say that a statement is true as \( x \) increases without bound means that for some (large) real number \( M \), the statement is true for all \( x \) in the interval \( \{x : x > M\} \). The following definition uses this concept.

Definition of Limits at Infinity

Let \( L \) be a real number.

1. The statement \( \lim_{x \to \infty} f(x) = L \) means that for each \( \varepsilon > 0 \) there exists an \( M > 0 \) such that \( |f(x) - L| < \varepsilon \) whenever \( x > M \).

2. The statement \( \lim_{x \to -\infty} f(x) = L \) means that for each \( \varepsilon > 0 \) there exists an \( N < 0 \) such that \( |f(x) - L| < \varepsilon \) whenever \( x < N \).

The definition of a limit at infinity is shown in Figure 3.34. In this figure, note that for a given positive number \( \varepsilon \) there exists a positive number \( M \) such that, for \( x > M \), the graph of \( f \) will lie between the horizontal lines given by \( y = L + \varepsilon \) and \( y = L - \varepsilon \).
SECTION 3.5  Limits at Infinity

Horizontal Asymptotes
In Figure 3.34, the graph of \( f \) approaches the line \( y = L \) as \( x \) increases without bound. The line \( y = L \) is called a\( \quad \) horizontal asymptote \( \) of the graph of \( f \).

**Definition of a Horizontal Asymptote**

The line \( y = L \) is a\( \quad \) horizontal asymptote \( \) of the graph of \( f \) if

\[
\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.
\]

Note that from this definition, it follows that the graph of a \( \textit{function} \) of \( x \) can have at most two horizontal asymptotes—one to the right and one to the left.

Limits at infinity have many of the same properties of limits discussed in Section 1.3. For example, if \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to \infty} g(x) \) both exist, then

\[
\lim_{x \to \infty} [f(x) + g(x)] = \lim_{x \to \infty} f(x) + \lim_{x \to \infty} g(x)
\]

and

\[
\lim_{x \to \infty} [f(x)g(x)] = \left[ \lim_{x \to \infty} f(x) \right] \left[ \lim_{x \to \infty} g(x) \right].
\]

Similar properties hold for limits at \( -\infty \).

When evaluating limits at infinity, the following theorem is helpful. (A proof of this theorem is given in Appendix A.)

**Theorem 3.10  Limits at Infinity**

If \( r \) is a positive rational number and \( c \) is any real number, then

\[
\lim_{x \to \infty} \frac{c}{x^r} = 0.
\]

Furthermore, if \( x^r \) is defined when \( x < 0 \), then

\[
\lim_{x \to -\infty} \frac{c}{x^r} = 0.
\]

**Example 1  Finding a Limit at Infinity**

Find the limit: \( \lim_{x \to \infty} \left( 5 - \frac{2}{x^2} \right) \).

**Solution**  Using Theorem 3.10, you can write

\[
\lim_{x \to \infty} \left( 5 - \frac{2}{x^2} \right) = \lim_{x \to \infty} 5 - \lim_{x \to \infty} \frac{2}{x^2} \quad \text{Property of limits}
\]

\[
= 5 - 0
\]

\[
= 5.
\]
EXAMPLE 2  Finding a Limit at Infinity

Find the limit: \( \lim_{x \to \infty} \frac{2x - 1}{x + 1} \).

Solution  Note that both the numerator and the denominator approach infinity as \( x \) approaches infinity.

\[
\lim_{x \to \infty} \frac{2x - 1}{x + 1} = \lim_{x \to \infty} \frac{2x}{x} - \frac{1}{x + 1} = \lim_{x \to \infty} \frac{2x}{x} - \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}}
\]

This results in \( \frac{\infty}{\infty} \), an indeterminate form. To resolve this problem, you can divide both the numerator and the denominator by \( x \). After dividing, the limit may be evaluated as shown.

\[
\lim_{x \to \infty} \frac{2x}{x} - \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}} = \frac{2}{1} - \frac{0}{1} = 2
\]

So, the line \( y = 2 \) is a horizontal asymptote to the right. By taking the limit as \( x \to -\infty \), you can see that \( y = 2 \) is also a horizontal asymptote to the left. The graph of the function is shown in Figure 3.35.

TECHNOLOGY  You can test the reasonableness of the limit found in Example 2 by evaluating \( f(x) \) for a few large positive values of \( x \). For instance, \( f(100) \approx 1.9703 \), \( f(1000) \approx 1.9970 \), and \( f(10,000) \approx 1.9997 \).

Another way to test the reasonableness of the limit is to use a graphing utility. For instance, in Figure 3.36, the graph of

\[
f(x) = \frac{2x - 1}{x + 1}
\]

is shown with the horizontal line \( y = 2 \). Note that as \( x \) increases, the graph of \( f \) moves closer and closer to its horizontal asymptote.
FOR FURTHER INFORMATION

For Figure 3.37

Website Teacher Patricia Clark Kenschaft in "Why Women Succeed in Mathematics" by Mona Fabricant, Sylvia Svitak, and Patricia Clark Kenschaft in Mathematics Teacher. To view this article, go to the website www.matharticles.com.

**EXAMPLE 3  A Comparison of Three Rational Functions**

Find each limit.

a. \( \lim_{x \to \infty} \frac{2x + 5}{3x^2 + 1} \)

b. \( \lim_{x \to \infty} \frac{2x^2 + 5}{3x^2 + 1} \)

c. \( \lim_{x \to \infty} \frac{2x^3 + 5}{3x^2 + 1} \)

**Solution** In each case, attempting to evaluate the limit produces the indeterminate form \( \infty/\infty \).

a. Divide both the numerator and the denominator by \( x^2 \).

\[
\lim_{x \to \infty} \frac{2x + 5}{3x^2 + 1} = \lim_{x \to \infty} \frac{(2/x) + (5/x^2)}{3 + (1/x^2)} = \frac{0 + 0}{3 + 0} = 0
\]

b. Divide both the numerator and the denominator by \( x^2 \).

\[
\lim_{x \to \infty} \frac{2x^2 + 5}{3x^2 + 1} = \lim_{x \to \infty} \frac{2 + (5/x^2)}{3 + (1/x^2)} = \frac{2 + 0}{3 + 0} = \frac{2}{3}
\]

c. Divide both the numerator and the denominator by \( x^2 \).

\[
\lim_{x \to \infty} \frac{2x^3 + 5}{3x^2 + 1} = \lim_{x \to \infty} \frac{2 + (5/x^2)}{3 + (1/x^2)} = \frac{\infty}{\infty}
\]

You can conclude that the limit *does not exist* because the numerator increases without bound while the denominator approaches 3.

**Guidelines for Finding Limits at \( \pm \infty \) of Rational Functions**

1. If the degree of the numerator is less than the degree of the denominator, then the limit of the rational function is 0.

2. If the degree of the numerator is equal to the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.

3. If the degree of the numerator is greater than the degree of the denominator, then the limit of the rational function does not exist.

Use these guidelines to check the results in Example 3. These limits seem reasonable when you consider that for large values of \( x \), the highest-power term of the rational function is the most “influential” in determining the limit. For instance, the limit as \( x \) approaches infinity of the function

\[
f(x) = \frac{1}{x^2 + 1}
\]

is 0 because the denominator overpowers the numerator as \( x \) increases or decreases without bound, as shown in Figure 3.37.

The function shown in Figure 3.37 is a special case of a type of curve studied by the Italian mathematician Maria Gaetana Agnesi. The general form of this function is

\[
f(x) = \frac{8a^3}{x^2 + 4a^2}
\]

and, through a mistranslation of the Italian word *verté*, the curve has come to be known as the Witch of Agnesi. Agnesi’s work with this curve first appeared in a comprehensive text on calculus that was published in 1748.
In Figure 3.37, you can see that the function \( f(x) = \frac{1}{x^2 + 1} \) approaches the same horizontal asymptote to the right and to the left. This is always true of rational functions. Functions that are not rational, however, may approach different horizontal asymptotes to the right and to the left. This is demonstrated in Example 4.

**EXAMPLE 4  A Function with Two Horizontal Asymptotes**

Find each limit.

a. \( \lim_{x \to \infty} \frac{3x - 2}{\sqrt{2x^2 + 1}} \)

**Solution**

a. For \( x > 0 \), you can write \( x = \sqrt{x^2} \). So, dividing both the numerator and the denominator by \( x \) produces

\[
\frac{3x - 2}{\sqrt{2x^2 + 1}} = \frac{3 - \frac{2}{x}}{\frac{1}{\sqrt{x^2}}} = 3 - \frac{2}{x},
\]

and you can take the limit as follows.

\[
\lim_{x \to \infty} \frac{3x - 2}{\sqrt{2x^2 + 1}} = \lim_{x \to \infty} 3 - \frac{2}{x} = 3 - 0 = 3
\]

b. For \( x < 0 \), you can write \( x = -\sqrt{x^2} \). So, dividing both the numerator and the denominator by \( x \) produces

\[
\frac{3x - 2}{\sqrt{2x^2 + 1}} = \frac{3x - 2}{\sqrt{2x^2 + 1}} = \frac{3 - \frac{2}{x}}{-\frac{1}{\sqrt{x^2}}} = 3 - \frac{2}{x},
\]

and you can take the limit as follows.

\[
\lim_{x \to -\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}} = \lim_{x \to -\infty} 3 - \frac{2}{x} = 3 - 0 = 3
\]

The graph of \( f(x) = (3x - 2)/\sqrt{2x^2 + 1} \) is shown in Figure 3.38.

**TECHNOLOGY PITFALL** If you use a graphing utility to help estimate a limit, be sure that you also confirm the estimate analytically—the pictures shown by a graphing utility can be misleading. For instance, Figure 3.39 shows one view of the graph of

\[
y = \frac{2x^3 + 1000x^2 + 1}{x^3 + 1000x^2 + x + 1000}.
\]

From this view, one could be convinced that the graph has \( y = 1 \) as a horizontal asymptote. An analytical approach shows that the horizontal asymptote is actually \( y = 2 \). Confirm this by enlarging the viewing window on the graphing utility.
In Section 1.3 (Example 9), you saw how the Squeeze Theorem can be used to evaluate limits involving trigonometric functions. This theorem is also valid for limits at infinity.

**EXAMPLE 5** Limits Involving Trigonometric Functions

Find each limit.

a. \( \lim_{x \to \infty} \sin x \)

b. \( \lim_{x \to \infty} \frac{\sin x}{x} \)

**Solution**

a. As \( x \) approaches infinity, the sine function oscillates between 1 and \(-1\). So, this limit does not exist.

b. Because \(-1 \leq \sin x \leq 1\), it follows that for \( x > 0 \),

\[
-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}
\]

where \( \lim_{x \to \infty} (-1/x) = 0 \) and \( \lim_{x \to \infty} (1/x) = 0 \). So, by the Squeeze Theorem, you can obtain

\[
\lim_{x \to \infty} \frac{\sin x}{x} = 0
\]

as shown in Figure 3.40.

**EXAMPLE 6** Oxygen Level in a Pond

Suppose that \( f(t) \) measures the level of oxygen in a pond, where \( f(t) = 1 \) is the normal (unpolluted) level and the time \( t \) is measured in weeks. When \( t = 0 \), organic waste is dumped into the pond, and as the waste material oxidizes, the level of oxygen in the pond is

\[
f(t) = \frac{t^2 - t + 1}{t^2 + 1}.
\]

What percent of the normal level of oxygen exists in the pond after 1 week? After 2 weeks? After 10 weeks? What is the limit as \( t \) approaches infinity?

**Solution**

When \( t = 1, 2, \) and 10, the levels of oxygen are as shown.

\[
f(1) = \frac{1^2 - 1 + 1}{1^2 + 1} = \frac{1}{2} = 50\% \quad \text{1 week}
\]

\[
f(2) = \frac{2^2 - 2 + 1}{2^2 + 1} = \frac{3}{5} = 60\% \quad \text{2 weeks}
\]

\[
f(10) = \frac{10^2 - 10 + 1}{10^2 + 1} = \frac{91}{101} \approx 90.1\% \quad \text{10 weeks}
\]

To find the limit as \( t \) approaches infinity, divide the numerator and the denominator by \( t^2 \) to obtain

\[
\lim_{t \to \infty} \frac{t^2 - t + 1}{t^2 + 1} = \lim_{t \to \infty} \frac{1 - (1/t) + (1/t^2)}{1 + (1/t^2)} = \frac{1 - 0 + 0}{1 + 0} = 1 = 100\%.
\]

See Figure 3.41.
Infinite Limits at Infinity

Many functions do not approach a finite limit as \( x \) increases (or decreases) without bound. For instance, no polynomial function has a finite limit at infinity. The following definition is used to describe the behavior of polynomial and other functions at infinity.

### Definition of Infinite Limits at Infinity

Let \( f \) be a function defined on the interval \((a, \infty)\).

1. The statement \( \lim_{x \to \infty} f(x) = \infty \) means that for each positive number \( M \), there is a corresponding number \( N > 0 \) such that \( f(x) > M \) whenever \( x > N \).
2. The statement \( \lim_{x \to -\infty} f(x) = -\infty \) means that for each negative number \( M \), there is a corresponding number \( N > 0 \) such that \( f(x) < M \) whenever \( x > N \).

Similar definitions can be given for the statements \( \lim_{x \to \infty} f(x) = -\infty \) and \( \lim_{x \to -\infty} f(x) = \infty \).

### Example 7 Finding Infinite Limits at Infinity

Find each limit.

a. \( \lim_{x \to \infty} x^3 \)

b. \( \lim_{x \to -\infty} x^3 \)

**Solution**

a. As \( x \) increases without bound, \( x^3 \) also increases without bound. So, you can write \( \lim_{x \to \infty} x^3 = \infty \).

b. As \( x \) decreases without bound, \( x^3 \) also decreases without bound. So, you can write \( \lim_{x \to -\infty} x^3 = -\infty \).

The graph of \( f(x) = x^3 \) in Figure 3.42 illustrates these two results. These results agree with the Leading Coefficient Test for polynomial functions as described in Section P.3.

### Example 8 Finding Infinite Limits at Infinity

Find each limit.

a. \( \lim_{x \to \infty} \frac{2x^2 - 4x}{x + 1} \)

b. \( \lim_{x \to -\infty} \frac{2x^2 - 4x}{x + 1} \)

**Solution** One way to evaluate each of these limits is to use long division to rewrite the improper rational function as the sum of a polynomial and a rational function.

a. \( \lim_{x \to \infty} \frac{2x^2 - 4x}{x + 1} = \lim_{x \to \infty} \left( 2x - 6 + \frac{6}{x + 1} \right) = \infty \)

b. \( \lim_{x \to -\infty} \frac{2x^2 - 4x}{x + 1} = \lim_{x \to -\infty} \left( 2x - 6 + \frac{6}{x + 1} \right) = -\infty \)

The statements above can be interpreted as saying that as \( x \) approaches \( \pm \infty \), the function \( f(x) = (2x^2 - 4x)/(x + 1) \) behaves like the function \( g(x) = 2x - 6 \). In Section 3.6, you will see that this is graphically described by saying that the line \( y = 2x - 6 \) is a slant asymptote of the graph of \( f \), as shown in Figure 3.43.
Exercises for Section 3.5

In Exercises 1 and 2, describe in your own words what the statement means.

1. \( \lim_{x \to \infty} f(x) = 4 \)
2. \( \lim_{x \to -\infty} f(x) = 2 \)

In Exercises 3–8, match the function with one of the graphs [(a), (b), (c), (d), (e), or (f)] using horizontal asymptotes as an aid.

(a) \( y \)  
(b) \( y \)

(c) \( y \)  
(d) \( y \)

(e) \( y \)  
(f) \( y \)

3. \( f(x) = \frac{3x^2}{x^2 + 2} \)
4. \( f(x) = \frac{2x}{\sqrt{x^2 + 2}} \)
5. \( f(x) = \frac{x}{x^2 + 2} \)
6. \( f(x) = 2 + \frac{x^2}{x^2 + 1} \)
7. \( f(x) = \frac{4 \sin x}{x^3 + 1} \)
8. \( f(x) = \frac{2x^2 - 3x + 5}{x^2 + 1} \)

\[ \begin{array}{|c|cccccc|}
\hline
x & 10^0 & 10^1 & 10^2 & 10^3 & 10^4 & 10^5 & 10^6 \\
\hline
f(x) & & & & & & & \\
\hline
\end{array} \]

9. \( f(x) = \frac{4x + 3}{2x - 1} \)
10. \( f(x) = \frac{2x^2}{x + 1} \)

In Exercises 11 and 12, find \( f(x) = \frac{8x}{\sqrt{x^2 - 9}} \) if possible.

11. \( f(x) = \frac{-6x}{\sqrt{4x^2 + 5}} \)
12. \( f(x) = 4 + \frac{3}{x^2 + 2} \)

In Exercises 15 and 16, find \( \lim_{x \to \infty} h(x) \), if possible.

15. \( f(x) = 5x^3 - 3x^2 + 10 \)
(a) \( h(x) = \frac{f(x)}{x^2} \)
(b) \( h(x) = \frac{f(x)}{x^3} \)
(c) \( h(x) = \frac{f(x)}{x^4} \)
16. \( f(x) = 5x^2 - 3x + 7 \)
(a) \( h(x) = \frac{f(x)}{x} \)
(b) \( h(x) = \frac{f(x)}{x^2} \)
(c) \( h(x) = \frac{f(x)}{x^3} \)

In Exercises 17–20, find each limit, if possible.

17. \( \lim_{x \to \infty} \frac{x^2 + 2}{x^2 - 1} \)
(a) \( \lim_{x \to \infty} \frac{x^2 + 2}{x^2 - 1} \)
(b) \( \lim_{x \to \infty} \frac{x^2 + 2}{x^2 - 1} \)
(c) \( \lim_{x \to \infty} \frac{x^2 + 2}{x^2 - 1} \)
18. \( \lim_{x \to \infty} \frac{x^3 + 2}{3x^3 - 1} \)
(a) \( \lim_{x \to \infty} \frac{3 - 2x}{3x^3 - 1} \)
(b) \( \lim_{x \to \infty} \frac{3 - 2x}{3x^3 - 1} \)
(c) \( \lim_{x \to \infty} \frac{3 - 2x}{3x^3 - 1} \)
19. \( \lim_{x \to \infty} \frac{5 - 2x^{1/2}}{3x^2 - 4} \)
(a) \( \lim_{x \to \infty} \frac{5 - 2x^{1/2}}{3x^2 - 4} \)
(b) \( \lim_{x \to \infty} \frac{5 - 2x^{1/2}}{3x^2 - 4} \)
(c) \( \lim_{x \to \infty} \frac{5 - 2x^{1/2}}{3x^2 - 4} \)
20. \( \lim_{x \to \infty} \frac{5x^{3/2}}{4x^{3/2} + 1} \)
(a) \( \lim_{x \to \infty} \frac{5x^{3/2}}{4x^{3/2} + 1} \)
(b) \( \lim_{x \to \infty} \frac{5x^{3/2}}{4x^{3/2} + 1} \)
(c) \( \lim_{x \to \infty} \frac{5x^{3/2}}{4x^{3/2} + 1} \)

In Exercises 21–34, find the limit.

21. \( \lim_{x \to \infty} \frac{2x - 1}{3x + 2} \)
22. \( \lim_{x \to \infty} \frac{3x^3 + 2}{9x^3 - 2x^2 + 7} \)
23. \( \lim_{x \to \infty} \frac{x}{x^2 - 1} \)
24. \( \lim_{x \to \infty} \left( \frac{4 + \frac{3}{x}}{x} \right) \)
25. \( \lim_{x \to \infty} \frac{3x^2}{x + 3} \)
26. \( \lim_{x \to \infty} \left( \frac{1}{2} - \frac{4}{x^2} \right) \)
27. \( \lim_{x \to \infty} \frac{x}{\sqrt{x^2 - x}} \)
28. \( \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} \)
29. \( \lim_{x \to \infty} \frac{2x + 1}{\sqrt{x^2 - x}} \)
30. \( \lim_{x \to \infty} \frac{-3x + 1}{\sqrt{x^2 + x}} \)
31. \( \lim_{x \to \infty} \frac{\sin 2x}{x} \)
32. \( \lim_{x \to \infty} \frac{x - \cos x}{x} \)
33. \( \lim_{x \to \infty} \frac{1}{2x + \sin x} \)
34. \( \lim_{x \to \infty} \cos \frac{1}{x} \)

Numerical and Graphical Analysis In Exercises 9–14, use a graphing utility to complete the table and estimate the limit as \( x \) approaches infinity. Then use a graphing utility to graph the function and estimate the limit graphically.
In Exercises 35–38, use a graphing utility to graph the function and identify any horizontal asymptotes.

35. \( f(x) = \frac{|x|}{x+1} \)  
36. \( f(x) = \frac{3x+2}{x-2} \)  
37. \( f(x) = \frac{3x}{\sqrt{x^2+2}} \)  
38. \( f(x) = \frac{\sqrt{9x^2+7}}{2x+1} \)

In Exercises 39 and 40, find the limit. (Hint: Let \( x = 1/t \) and find the limit as \( t \to 0^+ \).)

39. \( \lim_{x \to \infty} x \sin \frac{1}{x} \)  
40. \( \lim_{x \to \infty} x \tan \frac{1}{x} \)

In Exercises 41–46, find the limit. (Hint: Treat the expression as a fraction whose denominator is 1, and rationalize the numerator.) Use a graphing utility to verify your result.

41. \( \lim_{x \to \infty} \left( x + \sqrt{x^2 + 3} \right) \)  
42. \( \lim_{x \to \infty} \left( 2x - \sqrt{4x^2 + 1} \right) \)  
43. \( \lim_{x \to \infty} \left( x - \sqrt{x^2 + x} \right) \)  
44. \( \lim_{x \to \infty} \left( 3x + \sqrt{9x^2 - x} \right) \)  
45. \( \lim_{x \to \infty} \left( 4x - \sqrt{16x^2 - x} \right) \)  
46. \( \lim_{x \to \infty} \left( \frac{1}{x} + \frac{\sqrt{1}}{4x^2 + x} \right) \)

**Numerical, Graphical, and Analytic Analysis**  
In Exercises 47–50, use a graphing utility to complete the table and estimate the limit as \( x \) approaches infinity. Then use a graphing utility to graph the function and estimate the limit. Finally, find the limit analytically and compare your results with the estimates.

<table>
<thead>
<tr>
<th>( x )</th>
<th>10^9</th>
<th>10^8</th>
<th>10^7</th>
<th>10^6</th>
<th>10^5</th>
<th>10^4</th>
<th>10^3</th>
<th>10^2</th>
<th>10^1</th>
<th>10^0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

47. \( f(x) = x - \sqrt{x(x-1)} \)  
48. \( f(x) = x^2 - x\sqrt{x(x-1)} \)  
49. \( f(x) = x \sin \frac{1}{2x} \)  
50. \( f(x) = \frac{x + 1}{x\sqrt{x}} \)

**Writing About Concepts**

51. The graph of a function \( f \) is shown below. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

<table>
<thead>
<tr>
<th>( f(x) = 2 + x )</th>
<th>( y = \frac{x-3}{x-2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = \frac{x}{x^2 - 4} )</td>
<td>( y = \frac{2x}{9-x^2} )</td>
</tr>
<tr>
<td>( f(x) = x^2 + 9 )</td>
<td>( y = \frac{x^2}{x^2 - 9} )</td>
</tr>
<tr>
<td>( f(x) = \frac{x}{1-x} )</td>
<td>( y = \frac{2x}{x^2 + 4} )</td>
</tr>
<tr>
<td>( f(x) = 2 + \frac{3}{x^2} )</td>
<td>( y = 1 + \frac{1}{x} )</td>
</tr>
<tr>
<td>( f(x) = 3 + \frac{2}{x} )</td>
<td>( y = 4 \left( 1 - \frac{1}{x^2} \right) )</td>
</tr>
<tr>
<td>( f(x) = \frac{x^3}{\sqrt{x^2 - 4}} )</td>
<td>( y = \frac{x}{\sqrt{x^2 - 4}} )</td>
</tr>
</tbody>
</table>

52. Sketch a graph of a differentiable function \( f \) that satisfies the following conditions and has \( x = 2 \) as its only critical number.

- \( f(x) < 0 \) for \( x < 2 \)
- \( f(x) > 0 \) for \( x > 2 \)
- \( \lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 6 \)

53. Is it possible to sketch a graph of a function that satisfies the conditions of Exercise 52 and has no points of inflection? Explain.

54. If \( f \) is a continuous function such that \( \lim_{x \to \infty} f(x) = 5 \), find, if possible, \( \lim_{x \to \infty} f(x) \) for each specified condition.

(a) The graph of \( f \) is symmetric to the \( y \)-axis.
(b) The graph of \( f \) is symmetric to the origin.

In Exercises 55–72, sketch the graph of the function using extrema, intercepts, symmetry, and asymptotes. Then use a graphing utility to verify your result.

55. \( y = \frac{2+x}{1-x} \)  
56. \( y = \frac{x}{x^2 - 4} \)  
57. \( y = \frac{x}{x^2 - 4} \)  
58. \( y = \frac{2x}{9-x^2} \)  
59. \( y = x^2 + 9 \)  
60. \( y = \frac{x^2}{x^2 - 9} \)  
61. \( y = \frac{x^3}{\sqrt{x^2 - 4}} \)  
62. \( y = \frac{2x}{x^2 + 4} \)  
63. \( y = 4 \left( 1 - \frac{1}{x^2} \right) \)  
64. \( y = \frac{x}{1-x} \)  
65. \( y = \frac{2x}{9-x^2} \)  
66. \( y = \frac{x}{x^2 - 4} \)  
67. \( y = 2 + \frac{3}{x^2} \)  
68. \( y = 1 + \frac{1}{x} \)  
69. \( y = 3 + \frac{2}{x} \)  
70. \( y = 4 \left( 1 - \frac{1}{x^2} \right) \)  
71. \( y = \frac{x}{\sqrt{x^2 - 4}} \)  
72. \( y = \frac{x}{\sqrt{x^2 - 4}} \)

In Exercises 73–82, use a computer algebra system to analyze the graph of the function. Label any extrema and/or asymptotes that exist.

73. \( f(x) = 5 - \frac{1}{x^2} \)  
74. \( f(x) = \frac{x^2}{x^2 - 1} \)  
75. \( f(x) = \frac{x}{x^2 - 4} \)  
76. \( f(x) = \frac{1}{x^2 - x - 2} \)  
77. \( f(x) = \frac{x - 2}{x^2 - 4x + 3} \)  
78. \( f(x) = \frac{x + 1}{x^2 + x + 1} \)  
79. \( f(x) = \frac{3x}{\sqrt{3x^2 + 1}} \)  
80. \( g(x) = \frac{2x}{\sqrt{3x^2 + 1}} \)  
81. \( g(x) = \sin \left( \frac{x}{\sqrt{3}} \right), \ x > 3 \)  
82. \( f(x) = \frac{2 \sin 2x}{x} \)
In Exercises 83 and 84, (a) use a graphing utility to graph \( f \) and \( g \) in the same viewing window, (b) verify algebraically that \( f \) and \( g \) represent the same function, and (c) zoom out sufficiently far so that the graph appears as a line. What equation does this line appear to have? (Note that the points at which the function is not continuous are not readily seen when you zoom out.)

83. \( f(x) = \frac{x^3 - 3x^2 + 2}{x(x - 3)} \)

84. \( f(x) = -\frac{x^3 - 2x^2 + 2}{2x^2} \)

\( g(x) = x + \frac{2}{x(x - 3)} \)

85. **Average Cost**

A business has a cost of \( C = 0.5x + 500 \) for producing \( x \) units. The average cost per unit is

\[
\overline{C} = \frac{C}{x}
\]

Find the limit of \( \overline{C} \) as \( x \) approaches infinity.

86. **Engine Efficiency**

The efficiency of an internal combustion engine is

\[
\text{Efficiency (\%)} = 100 \left[ 1 - \frac{1}{(v_1/v_2)^N} \right]
\]

where \( v_1/v_2 \) is the ratio of the uncompressed gas to the compressed gas and \( c \) is a positive constant dependent on the engine design. Find the limit of the efficiency as the compression ratio approaches infinity.

87. **Physics**

Newton’s First Law of Motion and Einstein’s Special Theory of Relativity differ concerning a particle’s behavior as its velocity approaches the speed of light, \( c \). Functions \( N \) and \( E \) represent the predicted velocity, \( v \), with respect to time, \( t \), for a particle accelerated by a constant force. Write a limit statement that describes each theory.

88. **Temperature**

The graph shows the temperature \( T \), in degrees Fahrenheit, of an apple pie \( t \) seconds after it is removed from an oven and placed on a cooling rack.

(a) Find \( \lim_{t \to 0^+} T \). What does this limit represent?

(b) Find \( \lim_{t \to 90} T \). What does this limit represent?

89. **Modeling Data**

The table shows the world record times for running 1 mile, where \( t \) represents the year, with \( t = 0 \) corresponding to 1900, and \( y \) is the time in minutes and seconds.

<table>
<thead>
<tr>
<th>( t )</th>
<th>23</th>
<th>33</th>
<th>45</th>
<th>54</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>4:10.4</td>
<td>4:07.6</td>
<td>4:01.3</td>
<td>3:59.4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( t )</th>
<th>58</th>
<th>66</th>
<th>79</th>
<th>85</th>
<th>99</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>3:54.5</td>
<td>3:51.3</td>
<td>3:48.9</td>
<td>3:46.3</td>
<td>3:43.1</td>
</tr>
</tbody>
</table>

A model for the data is

\[
y = \frac{3.351r^2 + 42.461r - 543.730}{r^2}
\]

where the seconds have been changed to decimal parts of a minute.

(a) Use a graphing utility to plot the data and graph the model.

(b) Does there appear to be a limiting time for running 1 mile? Explain.

90. **Modeling Data**

The average typing speeds \( S \) (words per minute) of a typing student after \( t \) weeks of lessons are shown in the table.

<table>
<thead>
<tr>
<th>( t )</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S )</td>
<td>28</td>
<td>56</td>
<td>79</td>
<td>90</td>
<td>93</td>
<td>94</td>
</tr>
</tbody>
</table>

A model for the data is

\[
S = \frac{100r^2}{65 + r^2}, \quad r > 0.
\]

(a) Use a graphing utility to plot the data and graph the model.

(b) Does there appear to be a limiting typing speed? Explain.

91. **Modeling Data**

A heat probe is attached to the heat exchanger of a heating system. The temperature \( T \) (degrees Celsius) is recorded \( t \) seconds after the furnace is started. The results for the first 2 minutes are recorded in the table.

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>25.2°</td>
<td>36.9°</td>
<td>45.5°</td>
<td>51.4°</td>
<td>56.0°</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( t )</th>
<th>75</th>
<th>90</th>
<th>105</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>59.6°</td>
<td>62.0°</td>
<td>64.0°</td>
<td>65.2°</td>
</tr>
</tbody>
</table>

(a) Use the regression capabilities of a graphing utility to find a model of the form \( T_1 = at^2 + bt + c \) for the data.

(b) Use a graphing utility to graph \( T_1 \).

(c) A rational model for the data is \( T_2 = \frac{1451 + 86t}{58 + t} \). Use a graphing utility to graph the model.

(d) Find \( T_1(0) \) and \( T_2(0) \).

(e) Find \( \lim_{t \to 0^+} T_2 \).

(f) Interpret the result in part (e) in the context of the problem. Is it possible to do this type of analysis using \( T_1 \)? Explain.
92. Modeling Data  A container contains 5 liters of a 25\% brine solution. The table shows the concentrations $C$ of the mixture after adding $x$ liters of a 75\% brine solution to the container.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>0.25</td>
<td>0.295</td>
<td>0.333</td>
<td>0.365</td>
<td>0.393</td>
</tr>
</tbody>
</table>

(a) Use the regression features of a graphing utility to find a model of the form $C_1 = ax^2 + bx + c$ for the data.
(b) Use a graphing utility to graph $C_1$.
(c) A rational model for these data is $C_2 = \frac{5 + 3x}{20 + 4x}$. Use a graphing utility to graph $C_2$.
(d) Find $\lim_{x \to 2} C_1$ and $\lim_{x \to 2} C_2$. Which model do you think best represents the concentration of the mixture? Explain.
(e) What is the limiting concentration?

93. A line with slope $m$ passes through the point $(0, 4)$.
(a) Write the distance $d$ between the line and the point $(3, 1)$ as a function of $m$.
(b) Use a graphing utility to graph the equation in part (a).
(c) Find $\lim_{m \to \infty} d(m)$ and $\lim_{m \to -\infty} d(m)$. Interpret the results geometrically.

94. A line with slope $m$ passes through the point $(0, -2)$.
(a) Write the distance $d$ between the line and the point $(4, 2)$ as a function of $m$.
(b) Use a graphing utility to graph the equation in part (a).
(c) Find $\lim_{m \to \infty} d(m)$ and $\lim_{m \to -\infty} d(m)$. Interpret the results geometrically.

95. The graph of $f(x) = \frac{6x}{\sqrt{x^2 + 2}}$ is shown.

(a) Find $L = \lim_{x \to \infty} f(x)$.
(b) Determine $x_1$ and $x_2$ in terms of $e$.
(c) Determine $M$, where $M > 0$, such that $|f(x) - L| < \epsilon$ for $x > M$.
(d) Determine $N$, where $N < 0$, such that $|f(x) - L| < \epsilon$ for $x < N$.

96. The graph of $f(x) = \frac{6x}{\sqrt{x^2 + 2}}$ is shown.

(a) Find $L = \lim_{x \to \infty} f(x)$ and $K = \lim_{x \to -\infty} f(x)$.
(b) Determine $x_1$ and $x_2$ in terms of $e$.
(c) Determine $M$, where $M > 0$, such that $|f(x) - L| < \epsilon$ for $x > M$.
(d) Determine $N$, where $N < 0$, such that $|f(x) - K| < \epsilon$ for $x < N$.

97. Consider $\lim_{x \to \infty} \frac{3x}{\sqrt{x^2 + 3}}$. Use the definition of limits at infinity to find values of $M$ that correspond to (a) $\epsilon = 0.5$ and (b) $\epsilon = 0.1$.

98. Consider $\lim_{x \to \infty} \frac{3x}{\sqrt{x^2 + 3}}$. Use the definition of limits at infinity to find values of $N$ that correspond to (a) $\epsilon = 0.5$ and (b) $\epsilon = 0.1$.

In Exercises 99–102, use the definition of limits at infinity to prove the limit.

99. $\lim_{x \to \infty} \frac{1}{x} = 0$
100. $\lim_{x \to -\infty} \frac{2}{\sqrt{x}} = 0$
101. $\lim_{x \to \infty} \frac{1}{x^2} = 0$
102. $\lim_{x \to -\infty} \frac{1}{x^2} = 0$

103. Prove that if $p(x) = a_nx^n + \cdots + a_1x + a_0$ and $q(x) = b_mx^m + \cdots + b_1x + b_0 (a_n \neq 0, b_m \neq 0)$, then

$$\lim_{x \to \infty} \frac{p(x)}{q(x)} = \begin{cases} 0, & n < m \\ \frac{a_n}{b_m}, & n = m, \\ \pm \infty, & n > m \end{cases}$$

104. Use the definition of infinite limits at infinity to prove that $\lim_{x \to \infty} x^3 = \infty$.

True or False? In Exercises 105 and 106, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

105. If $f'(x) > 0$ for all real numbers $x$, then $f$ increases without bound.
106. If $f''(x) < 0$ for all real numbers $x$, then $f$ decreases without bound.