• Understand the definition of a Riemann sum.
• Evaluate a definite integral using limits.
• Evaluate a definite integral using properties of definite integrals.

Riemann Sums

In the definition of area given in Section 4.2, the partitions have subintervals of equal width. This was done only for computational convenience. The following example shows that it is not necessary to have subintervals of equal width.

**EXAMPLE 1  A Partition with Subintervals of Unequal Widths**

Consider the region bounded by the graph of \( f(x) = \sqrt{x} \) and the -axis for \( 0 \leq x \leq 1 \), as shown in Figure 4.18. Evaluate the limit

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x_i
\]

where \( c_i \) is the right endpoint of the partition given by \( c_i = \frac{i^2}{n^2} \) and \( \Delta x_i \) is the width of the \( i \)th interval.

**Solution**

The width of the \( i \)th interval is given by

\[
\Delta x_i = \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2} = \frac{i^2 - i^2 + 2i - 1}{n^2} = \frac{2i - 1}{n^2}.
\]

So, the limit is

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\frac{i^2}{n^2} \left( \frac{2i - 1}{n^2} \right)}
\]

\[
= \lim_{n \to \infty} \frac{1}{n^2} \sum_{i=1}^{n} (2i^2 - i)
\]

\[
= \lim_{n \to \infty} \frac{1}{n^2} \left[ \frac{n(n+1)(2n+1)}{6} \right] - \frac{n(n+1)}{2}
\]

\[
= \lim_{n \to \infty} \frac{4n^3 + 3n^2 - n}{6n^3}
\]

\[
= \frac{2}{3}.
\]

From Example 7 in Section 4.2, you know that the region shown in Figure 4.19 has an area of \( \frac{1}{3} \). Because the square bounded by \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \) has an area of 1, you can conclude that the area of the region shown in Figure 4.18 has an area of \( \frac{2}{3} \). This agrees with the limit found in Example 1, even though that example used a partition having subintervals of unequal widths. The reason this particular partition gave the proper area is that as \( n \) increases, the width of the largest subinterval approaches zero. This is a key feature of the development of definite integrals.
In the preceding section, the limit of a sum was used to define the area of a region in the plane. Finding area by this means is only one of applications involving the limit of a sum. A similar approach can be used to determine quantities as diverse as arc lengths, average values, centroids, volumes, work, and surface areas. The following definition is named after Georg Friedrich Bernhard Riemann. Although the definite integral had been defined and used long before the time of Riemann, he generalized the concept to cover a broader category of functions.

In the following definition of a Riemann sum, note that the function has no restrictions other than being defined on the interval $(a, b)$. (In the preceding section, the function was assumed to be continuous and nonnegative because we were dealing with the area under a curve.)

**Definition of a Riemann Sum**

Let $f$ be defined on the closed interval $[a, b]$, and let $\Delta$ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

where $\Delta x_i$ is the width of the $i$th subinterval. If $c_i$ is any point in the $i$th subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i) \Delta x_i, \quad x_{i-1} \leq c_i \leq x_i$$

is called a Riemann sum of $f$ for the partition $\Delta$.

**Regular partition**

For a general partition, the norm is related to the number of subintervals of $[a, b]$ in the following way.

$$\frac{b - a}{\|\Delta\|} \leq n$$

So, the number of subintervals in a partition approaches infinity as the norm of the partition approaches 0. That is, $\|\Delta\| \to 0$ implies that $n \to \infty$.

The converse of this statement is not true. For example, let $\Delta_n$ be the partition of the interval $[0, 1]$ given by

$$0 < \frac{1}{2^n} < \frac{1}{2^{n-1}} < \cdots < \frac{1}{8} < \frac{1}{4} < \frac{1}{2} < 1.$$  

As shown in Figure 4.20, for any positive value of $n$, the norm of the partition $\Delta_n$ is $\frac{1}{2^n}$.

So, letting $n$ approach infinity does not force $\|\Delta\|$ to approach 0. In a regular partition, however, the statements $\|\Delta\| \to 0$ and $n \to \infty$ are equivalent.
**Definite Integrals**

To define the definite integral, consider the following limit.

$$\lim_{|\Delta|\to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i = L$$

To say that this limit exists means that for there exists a such that for every partition with \(|\Delta| < \delta\) it follows that

$$|L - \sum_{i=1}^{n} f(c_i) \Delta x_i| < \varepsilon.$$  

(This must be true for any choice of \(c_i\) in the \(i\)th subinterval of \(\Delta\)).

**FOR FURTHER INFORMATION** For insight into the history of the definite integral, see the article “The Evolution of Integration” by A. Shenitzer and J. Steprans in *The American Mathematical Monthly*. To view this article, go to the website [www.matharticles.com](http://www.matharticles.com).

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**Definition of a Definite Integral**

If \(f\) is defined on the closed interval \([a, b]\) and the limit

$$\lim_{|\Delta|\to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i$$

exists (as described above), then \(f\) is **integrable** on \([a, b]\) and the limit is denoted by

$$\lim_{|\Delta|\to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i = \int_{a}^{b} f(x) \, dx.$$  

The limit is called the **definite integral** of \(f\) from \(a\) to \(b\). The number \(a\) is the **lower limit** of integration, and the number \(b\) is the **upper limit** of integration.

It is not a coincidence that the notation for definite integrals is similar to that used for indefinite integrals. You will see why in the next section when the Fundamental Theorem of Calculus is introduced. For now it is important to see that definite integrals and indefinite integrals are different identities. A definite integral is a **number**, whereas an indefinite integral is a **family of functions**.

A sufficient condition for a function \(f\) to be integrable on \([a, b]\) is that it is continuous on \([a, b]\). A proof of this theorem is beyond the scope of this text.

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**THEOREM 4.4 Continuity Implies Integrability**

If a function \(f\) is continuous on the closed interval \([a, b]\), then \(f\) is integrable on \([a, b]\).

---

**EXPLORATION**

The Converse of Theorem 4.4 Is the converse of Theorem 4.4 true? That is, if a function is integrable, does it have to be continuous? Explain your reasoning and give examples.

Describe the relationships among continuity, differentiability, and integrability. Which is the strongest condition? Which is the weakest? Which conditions imply other conditions?
EXAMPLE 2 Evaluating a Definite Integral as a Limit

Evaluate the definite integral $\int_{-2}^{1} 2x \, dx$.

Solution The function $f(x) = 2x$ is integrable on the interval $[-2, 1]$ because it is continuous on $[-2, 1]$. Moreover, the definition of integrability implies that any partition whose norm approaches 0 can be used to determine the limit. For computational convenience, define $\Delta x$ by subdividing $[-2, 1]$ into $n$ subintervals of equal width

$$\Delta x = \frac{b - a}{n} = \frac{3}{n}.$$ 

Choosing $c_i$ as the right endpoint of each subinterval produces

$$c_i = a + i\Delta x = -2 + \frac{3i}{n}.$$ 

So, the definite integral is given by

$$\int_{-2}^{1} 2x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x$$

$$= \lim_{n \to \infty} \frac{6}{n} \sum_{i=1}^{n} \left( -2 + \frac{3i}{n} \right) \left( \frac{3}{n} \right)$$

$$= \lim_{n \to \infty} \frac{6}{n} \left( -2n + \frac{3}{n} \left[ \frac{n(n + 1)}{2} \right] \right)$$

$$= \lim_{n \to \infty} \left( -12 + 9 + \frac{9}{n} \right)$$

$$= -3.$$ 

Because the definite integral is negative, it does not represent the area of the region shown in Figure 4.21. Definite integrals can be positive, negative, or zero. For a definite integral to be interpreted as an area (as defined in Section 4.2), the function $f$ must be continuous and nonnegative on $[a, b]$, as stated in the following theorem. (The proof of this theorem is straightforward—you simply use the definition of area given in Section 4.2.)

THEOREM 4.5 The Definite Integral as the Area of a Region

If $f$ is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of $f$, the $x$-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_{a}^{b} f(x) \, dx.$$ 

(See Figure 4.22.)
As an example of Theorem 4.5, consider the region bounded by the graph of
\[ f(x) = 4x - x^2 \]
and the x-axis, as shown in Figure 4.23. Because \( f \) is continuous and nonnegative on the closed interval \([0, 4]\), the area of the region is

\[ \text{Area} = \int_0^4 (4x - x^2) \, dx. \]

A straightforward technique for evaluating a definite integral such as this will be discussed in Section 4.4. For now, however, you can evaluate a definite integral in two ways—you can use the limit definition or you can check to see whether the definite integral represents the area of a common geometric region such as a rectangle, triangle, or semicircle.

**EXAMPLE 3  Areas of Common Geometric Figures**

Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

a. \( \int_1^4 \, dx \)  
   b. \( \int_0^3 (x + 2) \, dx \)  
   c. \( \int_{-2}^{2} \sqrt{4 - x^2} \, dx \)

**Solution**  A sketch of each region is shown in Figure 4.24.

a. This region is a rectangle of height 4 and width 2.
   \[ \int_1^4 \, dx = (\text{Area of rectangle}) = 4(2) = 8 \]

b. This region is a trapezoid with an altitude of 3 and parallel bases of lengths 2 and 5. The formula for the area of a trapezoid is \( \frac{1}{2}h(b_1 + b_2) \).
   \[ \int_0^3 (x + 2) \, dx = (\text{Area of trapezoid}) = \frac{1}{2}(3)(2 + 5) = \frac{21}{2} \]

c. This region is a semicircle of radius 2. The formula for the area of a semicircle is \( \frac{1}{2}\pi r^2 \).
   \[ \int_{-2}^{2} \sqrt{4 - x^2} \, dx = (\text{Area of semicircle}) = \frac{1}{2}\pi(2^2) = 2\pi \]
**Properties of Definite Integrals**

The definition of the definite integral of \( f \) on the interval \([a, b] \) specifies that \( a < b \). Now, however, it is convenient to extend the definition to cover cases in which \( a = b \) or \( a > b \). Geometrically, the following two definitions seem reasonable. For instance, it makes sense to define the area of a region of zero width and finite height to be 0.

### Definitions of Two Special Definite Integrals

1. If \( f \) is defined at \( x = a \), then we define
   \[
   \int_a^a f(x) \, dx = 0.
   \]

2. If \( f \) is integrable on \([a, b] \), then we define
   \[
   \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx.
   \]

### Example 4 Evaluating Definite Integrals

a. Because the sine function is defined at \( x = \pi \), and the upper and lower limits of integration are equal, you can write
   \[
   \int_0^\pi \sin x \, dx = 0.
   \]

b. The integral \( \int_0^3 (x + 2) \, dx \) is the same as that given in Example 3(b) except that the upper and lower limits are interchanged. Because the integral in Example 3(b) has a value of \( \frac{21}{2} \), you can write
   \[
   \int_0^3 (x + 2) \, dx = -\int_0^3 (x + 2) \, dx = -\frac{21}{2}.
   \]

In Figure 4.25, the larger region can be divided at \( x = c \) into two subregions whose intersection is a line segment. Because the line segment has zero area, it follows that the area of the larger region is equal to the sum of the areas of the two smaller regions.

### Theorem 4.6 Additive Interval Property

If \( f \) is integrable on the three closed intervals determined by \( a, b, \) and \( c \), then
\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.
\]

### Example 5 Using the Additive Interval Property

\[
\int_{-1}^1 |x| \, dx = \int_{-1}^0 -x \, dx + \int_0^1 x \, dx \quad \text{Theorem 4.6}
\]

\[
= \frac{1}{2} + \frac{1}{2} \quad \text{Area of a triangle}
\]

\[= 1 \]
Because the definite integral is defined as the limit of a sum, it inherits the properties of summation given at the top of page 260.

**THEOREM 4.7 Properties of Definite Integrals**

If \( f \) and \( g \) are integrable on \([a, b]\) and \( k \) is a constant, then the functions of \( kf \) and \( f \pm g \) are integrable on \([a, b]\), and

1. \[ \int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx \]

2. \[ \int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx. \]

Note that Property 2 of Theorem 4.7 can be extended to cover any finite number of functions. For example,

\[ \int_a^b [f(x) + g(x) + h(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx + \int_a^b h(x) \, dx. \]

**EXAMPLE 6 Evaluation of a Definite Integral**

Evaluate \( \int_1^3 (-x^2 + 4x - 3) \, dx \) using each of the following values.

\[ \int_1^3 x^2 \, dx = \frac{26}{3}, \quad \int_1^3 x \, dx = 4, \quad \int_1^3 dx = 2 \]

**Solution**

\[
\int_1^3 (-x^2 + 4x - 3) \, dx = \int_1^3 (-x^2) \, dx + \int_1^3 4x \, dx + \int_1^3 (-3) \, dx \\
= -\int_1^3 x^2 \, dx + 4\int_1^3 x \, dx - 3\int_1^3 dx \\
= -\left( \frac{26}{3} \right) + 4(4) - 3(2) \\
= \frac{4}{3}
\]

If \( f \) and \( g \) are continuous on the closed interval \([a, b]\) and \( 0 \leq f(x) \leq g(x) \) for \( a \leq x \leq b \), the following properties are true. First, the area of the region bounded by the graph of \( f \) and the \( x \)-axis (between \( a \) and \( b \)) must be nonnegative. Second, this area must be less than or equal to the area of the region bounded by the graph of \( g \) and the \( x \)-axis (between \( a \) and \( b \)), as shown in Figure 4.26. These two results are generalized in Theorem 4.8. (A proof of this theorem is given in Appendix A.)
THEOREM 4.8 Preservation of Inequality

1. If \( f \) is integrable and nonnegative on the closed interval \([a, b]\), then
   \[
   0 \leq \int_a^b f(x) \, dx.
   \]

2. If \( f \) and \( g \) are integrable on the closed interval \([a, b]\) and \( f(x) \leq g(x) \) for every \( x \) in \([a, b]\), then
   \[
   \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.
   \]

---

E x e r c i s e s  f o r  S e c t i o n  4.3

In Exercises 1 and 2, use Example 1 as a model to evaluate the limit

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x_i
\]

over the region bounded by the graphs of the equations.

1. \( f(x) = \sqrt{x}, \ y = 0, \ x = 0, \ x = 3 \)  
   (Hint: Let \( c_i = 3i^2/n^2 \).)

2. \( f(x) = \sqrt{x}, \ y = 0, \ x = 0, \ x = 1 \)  
   (Hint: Let \( c_i = i^3/n^3 \).)

In Exercises 3–8, evaluate the definite integral by the limit definition.

3. \( \int_{1}^{10} 6 \, dx \)
4. \( \int_{-2}^{3} x \, dx \)
5. \( \int_{-1}^{1} x^3 \, dx \)
6. \( \int_{1}^{3} 3x^2 \, dx \)
7. \( \int_{2}^{1} (x^2 + 1) \, dx \)
8. \( \int_{-1}^{2} (3x^2 + 2) \, dx \)

In Exercises 9–12, write the limit as a definite integral on the interval \([a, b]\), where \( c_i \) is any point in the \( i \)th subinterval.

<table>
<thead>
<tr>
<th>Limit</th>
<th>Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>9. ( \lim_{n \to \infty} \sum_{i=1}^{n} (3c_i + 10) \Delta x_i )</td>
<td>([-1, 5])</td>
</tr>
<tr>
<td>10. ( \lim_{n \to \infty} \sum_{i=1}^{n} 6c_i (4 - c_i)^2 \Delta x_i )</td>
<td>([0, 4])</td>
</tr>
<tr>
<td>11. ( \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{c_i^4 + 4} \Delta x_i )</td>
<td>([0, 3])</td>
</tr>
<tr>
<td>12. ( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{3}{c_i^2} \Delta x_i )</td>
<td>([1, 3])</td>
</tr>
</tbody>
</table>

In Exercises 13–22, set up a definite integral that yields the area of the region. (Do not evaluate the integral.)

13. \( f(x) = 3 \)
14. \( f(x) = 4 - 2x \)
15. \( f(x) = 4 - |x| \)
16. \( f(x) = x^2 \)
17. \( f(x) = 4 - x^2 \)
18. \( f(x) = \frac{1}{x^2 + 1} \)
39. $\int_{0}^{\pi} (4x^3 - 3x + 2) \, dx$
40. $\int_{0}^{\pi} (6x + x^3) \, dx$

41. Given $\int_{0}^{5} f(x) \, dx = 10$ and $\int_{5}^{7} f(x) \, dx = 3$, evaluate
   
   (a) $\int_{0}^{5} f(x) \, dx$
   (b) $\int_{5}^{7} f(x) \, dx$
   (c) $\int_{5}^{7} 3f(x) \, dx$

42. Given $\int_{0}^{4} f(x) \, dx = 4$ and $\int_{4}^{6} f(x) \, dx = -1$, evaluate
   
   (a) $\int_{0}^{4} f(x) \, dx$
   (b) $\int_{4}^{6} f(x) \, dx$
   (c) $\int_{4}^{6} 5f(x) \, dx$

43. Given $\int_{2}^{6} f(x) \, dx = 10$ and $\int_{2}^{6} g(x) \, dx = -2$, evaluate
   
   (a) $\int_{2}^{6} (f(x) + g(x)) \, dx$
   (b) $\int_{2}^{6} (g(x) - f(x)) \, dx$
   (c) $\int_{2}^{6} 2g(x) \, dx$
   (d) $\int_{2}^{6} 3f(x) \, dx$

44. Given $\int_{-1}^{1} f(x) \, dx = 0$ and $\int_{0}^{\infty} f(x) \, dx = 5$, evaluate
   
   (a) $\int_{-1}^{1} f(x) \, dx$
   (b) $\int_{0}^{\infty} f(x) \, dx - \int_{-1}^{1} f(x) \, dx$
   (c) $\int_{-1}^{1} 3f(x) \, dx$
   (d) $\int_{0}^{\infty} 3f(x) \, dx$

45. Use the table of values to find lower and upper estimates of $\int_{0}^{10} f(x) \, dx$.
   Assume that $f$ is a decreasing function.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>32</td>
<td>24</td>
<td>12</td>
<td>-4</td>
<td>-20</td>
<td>-36</td>
</tr>
</tbody>
</table>

46. Use the table of values to estimate $\int_{0}^{6} f(x) \, dx$.

   Use three equal subintervals and the (a) left endpoints, (b) right endpoints, and (c) midpoints. If $f$ is an increasing function, how does each estimate compare with the actual value? Explain your reasoning.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>-6</td>
<td>0</td>
<td>8</td>
<td>18</td>
<td>30</td>
<td>50</td>
<td>80</td>
</tr>
</tbody>
</table>
47. **Think About It** The graph of $f$ consists of line segments and a semicircle, as shown in the figure. Evaluate each definite integral by using geometric formulas.

![Graph of f consisting of line segments and a semicircle](image)

(a) \[ \int_{-4}^{3} f(x) \, dx \]
(b) \[ \int_{2}^{6} f(x) \, dx \]
(c) \[ \int_{-4}^{2} f(x) \, dx \]
(d) \[ \int_{-4}^{6} f(x) \, dx \]
(e) \[ \int_{-4}^{2} |f(x)| \, dx \]
(f) \[ \int_{-4}^{6} [f(x) + 2] \, dx \]

48. **Think About It** The graph of $f$ consists of line segments, as shown in the figure. Evaluate each definite integral by using geometric formulas.

![Graph of f consisting of line segments](image)

(a) \[ \int_{0}^{1} -f(x) \, dx \]
(b) \[ \int_{3}^{4} 3f(x) \, dx \]
(c) \[ \int_{0}^{7} f(x) \, dx \]
(d) \[ \int_{5}^{11} f(x) \, dx \]
(e) \[ \int_{4}^{11} f(x) \, dx \]
(f) \[ \int_{4}^{11} f(x) \, dx \]

49. **Think About It** Consider the function $f$ that is continuous on the interval $[-5, 5]$ and for which \[ \int_{-5}^{5} f(x) \, dx = 4. \] Evaluate each integral.

(a) \[ \int_{-5}^{5} [f(x) + 2] \, dx \]
(b) \[ \int_{-2}^{3} f(x) + 2 \, dx \]
(c) \[ \int_{-5}^{5} f(x) \, dx \] (f is even.)
(d) \[ \int_{-5}^{5} f(x) \, dx \] (f is odd.)

50. **Think About It** A function $f$ is defined below. Use geometric formulas to find $\int_{0}^{4} f(x) \, dx$.

\[ f(x) = \begin{cases} 4, & x < 4 \\ x, & x \geq 4 \end{cases} \]

Writing About Concepts

In Exercises 51 and 52, use the figure to fill in the blank with the symbol $<, >, \text{or} \ =$.

![Figure with integral](image)

51. The interval $[1, 5]$ is partitioned into $n$ subintervals of equal width $\Delta x$, and $x_i$ is the left endpoint of the $i$th subinterval.

\[ \sum_{i=1}^{n} f(x_i) \Delta x \quad \int_{1}^{5} f(x) \, dx \]

52. The interval $[1, 5]$ is partitioned into $n$ subintervals of equal width $\Delta x$, and $x_i$ is the right endpoint of the $i$th subinterval.

\[ \sum_{i=1}^{n} f(x_i) \Delta x \quad \int_{1}^{5} f(x) \, dx \]

53. Determine whether the function $f(x) = \frac{1}{x - 4}$ is integrable on the interval $[3, 5]$. Explain.

54. Give an example of a function that is integrable on the interval $[-1, 1]$, but not continuous on $[-1, 1]$.

In Exercises 55–58, determine which value best approximates the definite integral. Make your selection on the basis of a sketch.

55. \[ \int_{0}^{4} \sqrt{x} \, dx \]
   (a) 5  (b) 3  (c) 10  (d) 2  (e) 8

56. \[ \int_{0}^{\pi/2} 4 \cos \pi x \, dx \]
   (a) 4  (b) $\frac{4}{3}$  (c) 16  (d) $2\pi$  (e) $-6$

57. \[ \int_{0}^{2} 2 \sin \pi x \, dx \]
   (a) 6  (b) $\frac{5}{2}$  (c) 4  (d) $\frac{3}{2}$

58. \[ \int_{0}^{9} (1 + \sqrt{x}) \, dx \]
   (a) $-3$  (b) 9  (c) 27  (d) 3
In Exercises 63–68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

63. \[ \int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \]

64. \[ \int_a^b f(x)g(x) \, dx = \left[ \int_a^b f(x) \, dx \right] \left[ \int_a^b g(x) \, dx \right] \]

65. If the norm of a partition approaches zero, then the number of subintervals approaches infinity.

66. If \( f \) is increasing on \([a, b]\), then the minimum value of \( f(x) \) on \([a, b]\) is \( f(a) \).

67. The value of \( \int_a^b f(x) \, dx \) must be positive.

68. The value of \( \int_2^5 \sin(c^2) \, dx \) is 0.

69. Find the Riemann sum for \( f(x) = x^2 + 3x \) over the interval \([0, 8]\), where \( x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 7, \) and \( x_4 = 8 \), and where \( c_1 = 1, c_2 = 2, c_3 = 5, \) and \( c_4 = 8 \).

70. Find the Riemann sum for \( f(x) = \sin x \) over the interval \([0, 2\pi]\), where \( x_0 = 0, x_1 = \pi/4, x_2 = \pi/3, x_3 = \pi, \) and \( x_4 = 2\pi, \) and where \( c_1 = \pi/6, c_2 = \pi/3, c_3 = 2\pi/3, \) and \( c_4 = 3\pi/2 \).

71. Prove that \( \int_a^b x \, dx = \frac{b^2 - a^2}{2} \).

72. Prove that \( \int_a^b x^2 \, dx = \frac{b^3 - a^3}{3} \).

73. Think About It Determine whether the Dirichlet function \( f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases} \)

is integrable on the interval \([0, 1]\). Explain.

74. Suppose the function \( f \) is defined on \([0, 1]\), as shown in the figure.

\[
\begin{cases}
0, & x = 0 \\
\frac{1}{x}, & 0 < x \leq 1
\end{cases}
\]

Show that \( \int_0^1 f(x) \, dx \) does not exist. Why doesn’t this contradict Theorem 4.4?

75. Find the constants \( a \) and \( b \) that maximize the value of \( \int_a^b (1 - x^2) \, dx \).

Explain your reasoning.

76. Evaluate, if possible, the integral \( \int_0^1 \lfloor x \rfloor \, dx \).

77. Determine \( \lim_{n \to \infty} \frac{1}{n}[1^2 + 2^2 + 3^2 + \cdots + n^2] \) by using an appropriate Riemann sum.