Section 5.1

The Natural Logarithmic Function: Differentiation

- Develop and use properties of the natural logarithmic function.
- Understand the definition of the number $e$.
- Find derivatives of functions involving the natural logarithmic function.

The Natural Logarithmic Function

Recall that the General Power Rule

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

has an important disclaimer—it doesn’t apply when $n = -1$. Consequently, you have not yet found an antiderivative for the function $f(x) = 1/x$. In this section, you will use the Second Fundamental Theorem of Calculus to define such a function. This antiderivative is a function that you have not encountered previously in the text. It is neither algebraic nor trigonometric, but falls into a new class of functions called logarithmic functions. This particular function is the natural logarithmic function.

Definition of the Natural Logarithmic Function

The natural logarithmic function is defined by

$$\ln x = \int_1^x \frac{1}{t} \, dt, \quad x > 0.$$  

The domain of the natural logarithmic function is the set of all positive real numbers.

From the definition, you can see that $\ln x$ is positive for $x > 1$ and negative for $0 < x < 1$, as shown in Figure 5.1. Moreover, $\ln(1) = 0$, because the upper and lower limits of integration are equal when $x = 1$.

EXPLORATION

Graphing the Natural Logarithmic Function  
Using only the definition of the natural logarithmic function, sketch a graph of the function. Explain your reasoning.

Figure 5.1
To sketch the graph of \( y = \ln x \), you can think of the natural logarithmic function as an antiderivative given by the differential equation

\[
\frac{dy}{dx} = \frac{1}{x}
\]

Figure 5.2 is a computer-generated graph, called a slope (or direction) field, showing small line segments of slope \( 1/x \). The graph of \( y = \ln x \) is the solution that passes through the point \((1, 0)\). You will study slope fields in Section 6.1.

The following theorem lists some basic properties of the natural logarithmic function.

**THEOREM 5.1 Properties of the Natural Logarithmic Function**

The natural logarithmic function has the following properties.

1. The domain is \((0, \infty)\) and the range is \((-\infty, \infty)\).
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

**Proof** The domain of \( f(x) = \ln x \) is \((0, \infty)\) by definition. Moreover, the function is continuous because it is differentiable. It is increasing because its derivative

\[
f'(x) = \frac{1}{x}
\]

is positive for \( x > 0 \), as shown in Figure 5.3. It is concave downward because

\[
f''(x) = -\frac{1}{x^2}
\]

is negative for \( x > 0 \). The proof that \( f \) is one-to-one is left as an exercise (see Exercise 111). The following limits imply that its range is the entire real line.

\[
\lim_{x \to 1^+} \ln x = -\infty \quad \text{and} \quad \lim_{x \to \infty} \ln x = \infty
\]

Verification of these two limits is given in Appendix A.

Using the definition of the natural logarithmic function, you can prove several important properties involving operations with natural logarithms. If you are already familiar with logarithms, you will recognize that these properties are characteristic of all logarithms.

**THEOREM 5.2 Logarithmic Properties**

If \( a \) and \( b \) are positive numbers and \( n \) is rational, then the following properties are true.

1. \( \ln(1) = 0 \)
2. \( \ln(ab) = \ln a + \ln b \)
3. \( \ln(a^n) = n \ln a \)
4. \( \ln \left( \frac{a}{b} \right) = \ln a - \ln b \)

Logarithms
Napier coined the term logarithm, from the two Greek words logos (or ratio) and arithmos (or number), to describe the theory that he spent 20 years developing and that first appeared in the book *Mirifici Logarithmorum canonis descriptio* (A Description of the Marvelous Rule of Logarithms).
CHAPTER 5 Logarithmic, Exponential, and Other Transcendental Functions

**Proof** The first property has already been discussed. The proof of the second property follows from the fact that two antiderivatives of the same function differ at most by a constant. From the Second Fundamental Theorem of Calculus and the definition of the natural logarithmic function, you know that

\[
\frac{d}{dx} [\ln x] = \frac{d}{dx} \left[ \int_1^x \frac{1}{t} \, dt \right] = \frac{1}{x}
\]

So, consider the two derivatives

\[
\frac{d}{dx} [\ln(ax)] = \frac{a}{ax} = \frac{1}{x}
\]

and

\[
\frac{d}{dx} [\ln a + \ln x] = 0 + \frac{1}{x} = \frac{1}{x}
\]

Because \(\ln(ax)\) and \((\ln a + \ln x)\) are both antiderivatives of \(1/x\), they must differ at most by a constant.

\[
\ln(ax) = \ln a + \ln x + C
\]

By letting \(x = 1\), you can see that \(C = 0\). The third property can be proved similarly by comparing the derivatives of \(\ln(x^n)\) and \(n \ln x\). Finally, using the second and third properties, you can prove the fourth property.

\[
\ln \left( \frac{a}{b} \right) = \ln[a(b^{-1})] = \ln a + \ln(b^{-1}) = \ln a - \ln b
\]

Example 1 shows how logarithmic properties can be used to expand logarithmic expressions.

**Example 1** Expanding Logarithmic Expressions

a. \(\ln \frac{10}{9} \equiv \ln 10 - \ln 9\)  
   **Property 4**

b. \(\ln \sqrt{3x + 2} = \ln(3x + 2)^{1/2}\)  
   *Rewrite with rational exponent.*

   \[
   = \frac{1}{2} \ln(3x + 2)
   \]
   **Property 3**

c. \(\ln \frac{6x}{5} = \ln(6x) - \ln 5\)  
   **Property 4**

   \[
   = \ln 6 + \ln x - \ln 5
   \]
   **Property 2**

d. \(\ln \frac{(x^2 + 3)^2}{x^{\sqrt{x^2 + 1}} + 1} = \ln(x^2 + 3)^2 - \ln(x^{\sqrt{x^2 + 1}} + 1)\)

   \[
   = 2 \ln(x^2 + 3) - \ln(x^{\sqrt{x^2 + 1}} + 1)
   \]

   \[
   = 2 \ln(x^2 + 3) - \ln(x + \ln(x^2 + 1)^{1/3})
   \]

   \[
   = 2 \ln(x^2 + 3) - \ln x - \ln(x^2 + 1)^{1/3}
   \]

   \[
   = 2 \ln(x^2 + 3) - \ln x - \frac{1}{3} \ln(x^2 + 1)
   \]

When using the properties of logarithms to rewrite logarithmic functions, you must check to see whether the domain of the rewritten function is the same as the domain of the original. For instance, the domain of \(f(x) = \ln x^2\) is all real numbers except \(x = 0\), and the domain of \(g(x) = 2 \ln x\) is all positive real numbers. (See Figure 5.4.)
The Number $e$

It is likely that you have studied logarithms in an algebra course. There, without the benefit of calculus, logarithms would have been defined in terms of a base number. For example, common logarithms have a base of 10 and therefore $\log_{10} 10 = 1$. (You will learn more about this in Section 5.5.)

The base for the natural logarithm is defined using the fact that the natural logarithmic function is continuous, is one-to-one, and has a range of $(-\infty, \infty)$. So, there must be a unique real number $e$ such that

$$
\ln e = \int_{1}^{e} \frac{1}{t} \, dt = 1.
$$

$e$ is the base for the natural logarithm because $\ln e = 1$. 

**Figure 5.5**

The letter $e$ denotes the positive real number such that

$$
\ln e = \int_{1}^{e} \frac{1}{t} \, dt = 1.
$$

**Definition of $e$**

For further information To learn more about the number $e$, see the article “Unexpected Occurrences of the Number $e$” by Harris S. Shultz and Bill Leonard in Mathematics Magazine. To view this article, go to the website www.matharticles.com.

Once you know that $\ln e = 1$, you can use logarithmic properties to evaluate the natural logarithms of several other numbers. For example, by using the property

$$
\ln(e^n) = n \ln e
$$

you can evaluate $\ln(e^n)$ for various values of $n$, as shown in the table and in Figure 5.6.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\frac{1}{e^3} \approx 0.050$</th>
<th>$\frac{1}{e^2} \approx 0.135$</th>
<th>$\frac{1}{e} \approx 0.368$</th>
<th>$e^0 = 1$</th>
<th>$e \approx 2.718$</th>
<th>$e^2 \approx 7.389$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ln x$</td>
<td>$-3$</td>
<td>$-2$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

The logarithms shown in the table above are convenient because the $x$-values are integer powers of $e$. Most logarithmic expressions are, however, best evaluated with a calculator.

**Example 2** Evaluating Natural Logarithmic Expressions

a. $\ln 2 \approx 0.693$

b. $\ln 32 \approx 5.466$

c. $\ln 0.1 \approx -2.303$
The Derivative of the Natural Logarithmic Function

The derivative of the natural logarithmic function is given in Theorem 5.3. The first part of the theorem follows from the definition of the natural logarithmic function as an antiderivative. The second part of the theorem is simply the Chain Rule version of the first part.

**THEOREM 5.3 Derivative of the Natural Logarithmic Function**

Let $u$ be a differentiable function of $x$.

1. \( \frac{d}{dx} \ln x = \frac{1}{x} \), \( x > 0 \)
2. \( \frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx} = \frac{u'}{u} \), \( u > 0 \)

**EXAMPLE 3 Differentiation of Logarithmic Functions**

a. \( \frac{d}{dx} \left[ \ln(2x) \right] = \frac{u'}{u} = \frac{2}{2x} = \frac{1}{x} \) \( u = 2x \)

b. \( \frac{d}{dx} \left[ \ln(x^2 + 1) \right] = \frac{u'}{u} = \frac{2x}{x^2 + 1} \) \( u = x^2 + 1 \)

c. \( \frac{d}{dx} \left[ x \ln x \right] = x \left( \frac{d}{dx} \ln x \right) + \ln x \left( \frac{d}{dx} x \right) \) Product Rule

\[ = x \left( \frac{1}{x} \right) + \ln x(1) = 1 + \ln x \]

d. \( \frac{d}{dx} \left[ (\ln x)^3 \right] = 3(\ln x)^2 \frac{d}{dx} \ln x \) Chain Rule

\[ = 3(\ln x)^2 \frac{1}{x} \]

Napier used logarithmic properties to simplify calculations involving products, quotients, and powers. Of course, given the availability of calculators, there is now little need for this particular application of logarithms. However, there is great value in using logarithmic properties to simplify differentiation involving products, quotients, and powers.

**EXAMPLE 4 Logarithmic Properties as Aids to Differentiation**

Differentiate \( f(x) = \ln \sqrt{x + 1} \).

**Solution** Because

\[ f(x) = \ln \sqrt{x + 1} = \ln(x + 1)^{1/2} = \frac{1}{2} \ln(x + 1) \]

Rewrite before differentiating.

you can write

\[ f'(x) = \frac{1}{2} \left( \frac{1}{x + 1} \right) = \frac{1}{2(x + 1)} \]

Differentiate.
EXAMPLE 5  Logarithmic Properties as Aids to Differentiation

Differentiate \( f(x) = \ln \frac{x(x^2 + 1)^2}{\sqrt{2x^3 - 1}} \)

Solution

\[
\begin{align*}
f(x) &= \ln \frac{x(x^2 + 1)^2}{\sqrt{2x^3 - 1}} \\
&= \ln x + 2 \ln(x^2 + 1) - \frac{1}{2} \ln(2x^3 - 1) \\
f'(x) &= \frac{1}{x} + 2 \left( \frac{2x}{x^2 + 1} \right) - \frac{1}{2} \left( \frac{6x^2}{2x^3 - 1} \right) \\
&= \frac{1}{x} + \frac{4x}{x^2 + 1} - \frac{3x^2}{2x^3 - 1}
\end{align*}
\]

NOTE  In Examples 4 and 5, be sure you see the benefit of applying logarithmic properties before differentiating. Consider, for instance, the difficulty of direct differentiation of the function given in Example 5.

On occasion, it is convenient to use logarithms as aids in differentiating nonlogarithmic functions. This procedure is called logarithmic differentiation.

EXAMPLE 6  Logarithmic Differentiation

Find the derivative of

\[
y = \frac{(x - 2)^2}{\sqrt{x^2 + 1}} \quad x \neq 2.
\]

Solution  Note that \( y > 0 \) for all \( x \neq 2 \). So, \( \ln y \) is defined. Begin by taking the natural logarithm of each side of the equation. Then apply logarithmic properties and differentiate implicitly. Finally, solve for \( y' \).

\[
\begin{align*}
y &= \frac{(x - 2)^2}{\sqrt{x^2 + 1}} \quad x \neq 2 \\
\ln y &= \ln \frac{(x - 2)^2}{\sqrt{x^2 + 1}} \\
\ln y &= 2 \ln(x - 2) - \frac{1}{2} \ln(x^2 + 1) \\
y' &= \frac{2}{x - 2} - \frac{x}{x^2 + 1} \quad \text{Differentiate.} \\
&= \left( \frac{2}{x - 2} - \frac{x}{x^2 + 1} \right) \\
&= \left( \frac{x^2 + 2x + 2}{(x - 2)(x^2 + 1)} \right) \quad \text{Solve for } y'. \\
&= \frac{(x - 2)(x^2 + 2x + 2)}{(x^2 + 1)^{3/2}} \quad \text{Substitute for } y.
\end{align*}
\]
Because the natural logarithm is undefined for negative numbers, you will often encounter expressions of the form $\ln|u|$. The following theorem states that you can differentiate functions of the form $y = \ln|u|$ as if the absolute value sign were not present.

**THEOREM 5.4 Derivative Involving Absolute Value**

If $u$ is a differentiable function of $x$ such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}.$$  

**Proof** If $u > 0$, then $|u| = u$, and the result follows from Theorem 5.3. If $u < 0$, then $|u| = -u$, and you have

$$\frac{d}{dx}[\ln|u|] = \frac{d}{dx}[\ln(-u)] = \frac{-u'}{-u} = \frac{u'}{u}.$$  

**EXAMPLE 7** Derivative Involving Absolute Value

Find the derivative of

$$f(x) = \ln|\cos x|.$$  

**Solution** Using Theorem 5.4, let $u = \cos x$ and write

$$\frac{d}{dx}[\ln|\cos x|] = \frac{u'}{u} \quad \frac{d}{dx}[\ln|u|] = \frac{u'}{u}$$  

$$= \frac{-\sin x}{\cos x} \quad u = \cos x$$  

$$= -\tan x. \quad \text{Simplify.}$$  

**EXAMPLE 8** Finding Relative Extrema

Locate the relative extrema of

$$y = \ln(x^2 + 2x + 3).$$  

**Solution** Differentiating $y$, you obtain

$$\frac{dy}{dx} = \frac{2x + 2}{x^2 + 2x + 3}.$$  

Because $dy/dx = 0$ when $x = -1$, you can apply the First Derivative Test and conclude that the point $(-1, \ln 2)$ is a relative minimum. Because there are no other critical points, it follows that this is the only relative extremum (see Figure 5.7).
Exercises for Section 5.1

1. Complete the table below. Use a graphing utility and Simpson’s Rule with \( n = 10 \) to approximate the integral \( \int_{1}^{2} \frac{1}{t} \, dt \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.5</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(1/t) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. (a) Plot the points generated in Exercise 1 and connect them with a smooth curve. Compare the result with the graph of \( y = \ln x \).

(b) Use a graphing utility to graph \( y = \int_{1}^{2} \frac{1}{t} \, dt \) for \( 0.2 \leq x \leq 4 \). Compare the result with the graph of \( y = \ln x \).

In Exercises 3–6, use a graphing utility to evaluate the logarithm by (a) using the natural logarithm key and (b) using the integration capabilities to evaluate the integral \( \int_{0.1}^{0.5} f(t) \, dt \).

3. \( \ln 45 \) 4. \( \ln 8.3 \)

5. \( \ln 0.8 \) 6. \( \ln 0.6 \)

In Exercises 7–10, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

7. \( f(x) = \ln x + 2 \) 8. \( f(x) = -\ln x \)

9. \( f(x) = \ln(x - 1) \) 10. \( f(x) = -\ln(-x) \)

In Exercises 11–16, sketch the graph of the function and state its domain.

11. \( f(x) = 3 \ln x \) 12. \( f(x) = -2 \ln x \)

13. \( f(x) = \ln 2x \) 14. \( f(x) = \ln|x| \)

15. \( f(x) = \ln(x - 1) \) 16. \( g(x) = 2 + \ln x \)

In Exercises 17 and 18, use the properties of logarithms to approximate the indicated logarithms, given that \( \ln 2 = 0.6931 \) and \( \ln 3 = 1.0986 \).

17. (a) \( \ln 6 \) (b) \( \ln \frac{3}{2} \) (c) \( \ln 81 \) (d) \( \ln \sqrt{3} \)

18. (a) \( \ln 0.25 \) (b) \( \ln 24 \) (c) \( \ln \sqrt[3]{12} \) (d) \( \ln \frac{1}{2} \)

In Exercises 19–28, use the properties of logarithms to expand the logarithmic expression.

19. \( \ln \frac{y}{z} \) 20. \( \ln \sqrt{e^x} \)

21. \( \ln \frac{x}{y} \) 22. \( \ln(xy) \)

23. \( \ln \sqrt[3]{a^2 + 1} \) 24. \( \ln \sqrt{x^2 - 1} \)

25. \( \ln \left( \frac{x^2 - 1}{x^3} \right) \) 26. \( \ln(x^3) \)

27. \( \ln(z - 1)^2 \) 28. \( \ln \frac{1}{e} \)

In Exercises 29–34, write the expression as a logarithm of a single quantity.

29. \( \ln(x - 2) - \ln(x + 2) \) 30. \( 3 \ln x + 2 \ln y - 4 \ln z \)

31. \( \frac{1}{2} \left( 2 \ln(x + 3) + \ln x - \ln(x^2 - 1) \right) \) 32. \( 2[\ln x - \ln(x + 1) - \ln(x - 1)] \)

33. \( 2 \ln 3 - \frac{3}{2} \ln(x^2 + 1) \) 34. \( \frac{2}{3} \left( \ln(x^2 + 1) - \ln(x + 1) - \ln(x - 1) \right) \)

In Exercises 35 and 36, (a) verify that \( f = g \) by using a graphing utility to graph \( f \) and \( g \) in the same viewing window. (b) Then verify that \( f = g \) algebraically.

35. \( f(x) = \ln \frac{x^2}{4}, \ x > 0 \) \( g(x) = 2 \ln x - \ln 4 \)

36. \( f(x) = \ln \sqrt{x(x^2 + 1)}, \ g(x) = \frac{1}{2} [\ln x + \ln(x^2 + 1)] \)

In Exercises 37–40, find the limit.

37. \( \lim_{x \to 3} \ln(x - 3) \) 38. \( \lim_{x \to 6} \ln(6 - x) \)

39. \( \lim_{x \to 2} \ln[x^2(3 - x)] \) 40. \( \lim_{x \to 5} \ln \frac{1}{\sqrt{x - 4}} \)

In Exercises 41–44, find an equation of the tangent line to the graph of the logarithmic function at the point \((1, 0)\).

41. \( y = \ln x^3 \) 42. \( y = \ln x^{\sqrt{2}} \)
43. \( y = \ln x^2 \)

44. \( y = \ln x^{1/2} \)

In Exercises 45–70, find the derivative of the function.

45. \( g(x) = \ln x^2 \)

46. \( h(x) = \ln(2x^2 + 1) \)

47. \( y = (\ln x)^4 \)

48. \( y = x \ln x \)

49. \( y = \ln(\sqrt{x^2 - 1}) \)

50. \( y = \ln \sqrt{x^2 - 4} \)

51. \( f(x) = \ln \left( \frac{x}{x^2 + 1} \right) \)

52. \( f(x) = \ln \left( \frac{2x}{x + 3} \right) \)

53. \( g(t) = \ln \left( \frac{t}{t^2} \right) \)

54. \( h(t) = \ln t \)

55. \( y = \ln(\ln x^2) \)

56. \( y = \ln(\ln x) \)

57. \( y = \ln \left( \frac{x + 1}{x - 1} \right) \)

58. \( y = \ln \left( \frac{x - 1}{x + 1} \right) \)

59. \( f(x) = \ln \left( \frac{4 + x^2}{x} \right) \)

60. \( f(x) = \ln \left( x + \sqrt{4 + x^2} \right) \)

61. \( y = \frac{\ln x + 1}{x} + \ln \left( x + \sqrt{x^2 + 1} \right) \)

62. \( y = \frac{\ln x + 4}{2x^2} - \frac{1}{4} \ln \left( \frac{2 + x^2 + 4}{x} \right) \)

63. \( y = \ln \left( \sin x \right) \)

64. \( y = \ln \left( \csc x \right) \)

65. \( y = \ln \left( \frac{\cos x}{\cos x - 1} \right) \)

66. \( y = \ln \left( \sec x + \tan x \right) \)

67. \( y = \ln \left( \frac{1 + \sin x}{2 + \sin x} \right) \)

68. \( y = \ln \left( \cos^2 x \right) \)

69. \( f(x) = \int_{\pi/2}^{\pi/2} (t + 1) \, dt \)

70. \( g(x) = \int_{1}^{\pi} (t^2 + 3) \, dt \)

In Exercises 71–76, (a) find an equation of the tangent line to the graph of \( f \) at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the derivative feature of a graphing utility to confirm your results.

71. \( f(x) = 3x^2 - \ln x, \quad (1, 3) \)

72. \( f(x) = 4 - x^2 - \ln(x^2 + 1), \quad (0, 4) \)

73. \( f(x) = \ln \left( \sqrt{1 + \sin^2 x} \right), \quad \left( \frac{\pi}{4}, \ln \sqrt{\frac{3}{2}} \right) \)

74. \( f(x) = \ln(\ln x^2), \quad (1, 0) \)

75. \( f(x) = x^3 \ln x, \quad (1, 0) \)

76. \( f(x) = \ln x, \quad (1, 0) \)

In Exercises 77 and 78, use implicit differentiation to find \( \frac{dy}{dx} \).

77. \( x^2 - 3 \ln y + y^2 = 10 \)

78. \( \ln xy + 5x = 30 \)

In Exercises 79 and 80, use implicit differentiation to find an equation of the tangent line to the graph at the given point.

79. \( x + y - 1 = \ln(x^2 + y^2), \quad (1, 0) \)

80. \( y^2 + \ln(xy) = 2, \quad (e, 1) \)

In Exercises 81 and 82, show that the function is a solution of the differential equation.

81. \( y = 2 \ln x + 3 \quad xy'' + y' = 0 \)

82. \( y = x \ln x - 4x \quad x + y - xy' = 0 \)

In Exercises 83–88, locate any relative extrema and inflection points. Use a graphing utility to confirm your results.

83. \( y = \frac{x^2}{2} - \ln x \)

84. \( y = x - \ln x \)

85. \( y = x \ln x \)

86. \( y = \frac{\ln x}{x} \)

87. \( y = \frac{x}{\ln x} \)

88. \( y = x^2 \ln \frac{x}{4} \)

**Linear and Quadratic Approximations**

In Exercises 89 and 90, use a graphing utility to graph the function. Then graph \( P_1(x) = f(1) + f'(1)(x - 1) \) and \( P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2} f''(1)(x - 1)^2 \) in the same viewing window. Compare the values of \( f \), \( P_1 \), and \( P_2 \) and their first derivatives at \( x = 1 \).

89. \( f(x) = \ln x \)

90. \( f(x) = x \ln x \)

In Exercises 91 and 92, use Newton’s Method to approximate, to three decimal places, the \( x \)-coordinate of the point of intersection of the graphs of the two equations. Use a graphing utility to verify your result.

91. \( y = \ln x, \quad y = -x \)

92. \( y = \ln x, \quad y = 3 - x \)

In Exercises 93–98, use logarithmic differentiation to find \( \frac{dy}{dx} \).

93. \( y = x\sqrt{x^2 - 1} \)

94. \( y = (x - 1)(x - 2)(x - 3) \)

95. \( y = \frac{x^3}{(x - 1)^2} \)

96. \( y = \frac{\sqrt{x^2 - 1}}{x^2 + 1} \)

97. \( y = \frac{x(x - 1)^{3/2}}{\sqrt{x + 1}} \)

98. \( y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)} \)

**Writing About Concepts**

99. In your own words, state the properties of the natural logarithmic function.

100. Define the base for the natural logarithmic function.

101. Let \( f \) be a function that is positive and differentiable on the entire real line. Let \( g(x) = \ln f(x) \).

   (a) If \( g \) is increasing, must \( f \) be increasing? Explain.

   (b) If the graph of \( f \) is concave upward, must the graph of \( g \) be concave upward? Explain.
Writing About Concepts (continued)

102. Consider the function \( f(x) = x - 2 \ln x \) on \([1, 3]\).
   (a) Explain why Rolle’s Theorem (Section 3.2) does not apply.
   (b) Do you think the conclusion of Rolle’s Theorem is true for \( f \)? Explain.

103. \( \ln(x + 25) = \ln x + \ln 25 \)
104. If \( y = \ln x \), then \( y' = 1/x \).

105. Home Mortgage The term \( t \) (in years) of a $120,000 home mortgage at 10% interest can be approximated by
   \[
   t = \frac{5.315}{-6.7968 + \ln x}, \quad x > 1000
   \]
   where \( x \) is the monthly payment in dollars.
   (a) Use a graphing utility to graph the model.
   (b) Use the model to approximate the term of a home mortgage for which the monthly payment is $1167.41. What is the total amount paid?
   (c) Use a graphing utility to graph the model for which the monthly payment is $1068.45. What is the total amount paid?
   (d) Find the instantaneous rate of change of \( t \) with respect to \( x \) when \( x = 1167.41 \) and \( x = 1068.45 \).
   (e) Write a short paragraph describing the benefit of the higher monthly payment.

106. Sound Intensity The relationship between the number of decibels \( \beta \) and the intensity of a sound \( I \) in watts per centimeter squared is
   \[
   \beta = 10 \log_{10} \left( \frac{I}{10^{-16}} \right).
   \]
   Use the properties of logarithms to write the formula in simpler form, and determine the number of decibels of a sound with an intensity of \( 10^{-10} \) watts per square centimeter.

107. Modeling Data The table shows the temperature \( T \) (°F) at which water boils at selected pressures \( p \) (pounds per square inch). (Source: Standard Handbook of Mechanical Engineers)

\[
\begin{array}{c|c|c|c|c|c}
 p & 5 & 10 & 14.696 & 20 & \hline
 T & 162.24 & 193.21 & 212.00 & 227.96 & \\
 & 50 & 60 & 80 & 100 & \\
 & 250.33 & 267.25 & 292.71 & 312.03 & 327.81 & \\
\end{array}
\]

A model that approximates the data is
\[
T = 87.97 + 34.96 \ln p + 7.91 \sqrt{p}.
\]

108. Modeling Data The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is one atmosphere (1.033227 kilograms per square centimeter). The table shows the pressures \( p \) (in atmospheres) at selected altitudes \( h \) (in kilometers).

\[
\begin{array}{c|c|c|c|c|c}
 h & 0 & 5 & 10 & 15 & 20 & 25 \\
 p & 1 & 0.55 & 0.25 & 0.12 & 0.06 & 0.02 \\
\end{array}
\]

(a) Use a graphing utility to find a model of the form \( p = a + b \ln h \) for the data. Explain why the result is an error message.
(b) Use a graphing utility to find the logarithmic model \( h = a + b \ln p \) for the data.
(c) Use a graphing utility to plot the data and graph the model.
(d) Use the model to estimate the altitude when \( p = 0.75 \).
(e) Use the model to estimate the pressure when \( h = 13 \).
(f) Use the model to find the rate of change of pressure when \( h = 5 \) and \( h = 20 \). Interpret the results.

109. Tractrix A person walking along a dock drags a boat by a 10-meter rope. The boat travels along a path known as a tractrix (see figure). The equation of this path is
\[
y = 10 \ln \left( \frac{10 + \sqrt{100 - x^2}}{x} \right) - \sqrt{100 - x^2}.
\]

(a) Use a graphing utility to graph the function.
(b) What is the slope of this path when \( x = 5 \) and \( x = 9 \)?
(c) What does the slope of the path approach as \( x \to 10 \)?

110. Conjecture Use a graphing utility to graph \( f \) and \( g \) in the same viewing window and determine which is increasing at the greater rate for large values of \( x \). What can you conclude about the rate of growth of the natural logarithmic function? (a) \( f(x) = \ln x \), \( g(x) = \sqrt{x} \) (b) \( f(x) = \ln x \), \( g(x) = \sqrt{x} - 4 \ln x \)
111. Prove that the natural logarithmic function is one-to-one.

(a) Use a graphing utility to graph \( y = \sqrt{x} - 4 \ln x \).
(b) Use the graph to identify any relative minima and inflection points.
(c) Use calculus to verify your answer to part (b).