Section 5.8

Hyperbolic Functions

- Develop properties of hyperbolic functions.
- Differentiate and integrate hyperbolic functions.
- Develop properties of inverse hyperbolic functions.
- Differentiate and integrate functions involving inverse hyperbolic functions.

Hyperbolic Functions

In this section you will look briefly at a special class of exponential functions called hyperbolic functions. The name hyperbolic function arose from comparison of the area of a semicircular region, as shown in Figure 5.35, with the area of a region under a hyperbola, as shown in Figure 5.36. The integral for the semicircular region involves an inverse trigonometric (circular) function:

\[ \int_{-1}^{1} \sqrt{1 - x^2} \, dx = \frac{1}{2} \left[ x \sqrt{1 - x^2} + \arcsin x \right]_{-1}^{1} = \frac{\pi}{2} \approx 1.571. \]

The integral for the hyperbolic region involves an inverse hyperbolic function:

\[ \int_{-1}^{1} \sqrt{1 + x^2} \, dx = \frac{1}{2} \left[ x \sqrt{1 + x^2} + \sinh^{-1} x \right]_{-1}^{1} \approx 2.296. \]

This is only one of many ways in which the hyperbolic functions are similar to the trigonometric functions.

Definitions of the Hyperbolic Functions

\[
\begin{align*}
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\tanh x &= \frac{\sinh x}{\cosh x} \\
\coth x &= \frac{1}{\tanh x} \\
\csch x &= \frac{1}{\sinh x}, \quad x \neq 0 \\
\sech x &= \frac{1}{\cosh x} \\
\end{align*}
\]

NOTE  \( \sinh x \) is read as “the hyperbolic sine of \( x \),” \( \cosh x \) as “the hyperbolic cosine of \( x \),” and so on.

FOR FURTHER INFORMATION For more information on the development of hyperbolic functions, see the article “An Introduction to Hyperbolic Functions in Elementary Calculus” by Jerome Rosenthal in Mathematics Teacher. To view this article, go to the website www.matharticles.com.
The graphs of the six hyperbolic functions and their domains and ranges are shown in Figure 5.37. Note that the graph of \( \sinh x \) can be obtained by adding ordinates using the exponential functions \( f(x) = \frac{1}{2}e^x \) and \( g(x) = -\frac{1}{2}e^{-x} \). Likewise, the graph of \( \cosh x \) can be obtained by adding ordinates using the exponential functions \( f(x) = \frac{1}{2}e^x \) and \( h(x) = \frac{1}{2}e^{-x} \).

Many of the trigonometric identities have corresponding hyperbolic identities. For instance,

\[
\cosh^2 x - \sinh^2 x = \left( \frac{e^x + e^{-x}}{2} \right)^2 - \left( \frac{e^x - e^{-x}}{2} \right)^2 \\
= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\
= \frac{4}{4} \\
= 1
\]

and

\[
2 \sinh x \cosh x = 2 \left( \frac{e^x - e^{-x}}{2} \right) \left( \frac{e^x + e^{-x}}{2} \right) \\
= \frac{e^{2x} - e^{-2x}}{2} \\
= \sinh 2x.
\]
CHAPTER 5 Logarithmic, Exponential, and Other Transcendental Functions

Differentiation and Integration of Hyperbolic Functions

Because the hyperbolic functions are written in terms of \( e^x \) and \( e^{-x} \), you can easily derive rules for their derivatives. The following theorem lists these derivatives with the corresponding integration rules.

**THEOREM 5.18 Derivatives and Integrals of Hyperbolic Functions**

Let \( u \) be a differentiable function of \( x \).

\[
\begin{align*}
\frac{d}{dx} \sinh u &= (\cosh u)u' \\
\frac{d}{dx} \cosh u &= (\sinh u)u' \\
\frac{d}{dx} \tanh u &= (\sech^2 u)u' \\
\frac{d}{dx} \coth u &= -(\csch^2 u)u' \\
\frac{d}{dx} \sech u &= -(\csch u \tanh u)u' \\
\frac{d}{dx} \csch u &= -(\csch u \coth u)u'
\end{align*}
\]

\[
\begin{align*}
\int \cosh u \, du &= \sinh u + C \\
\int \sinh u \, du &= \cosh u + C \\
\int \sech^2 u \, du &= \tanh u + C \\
\int \csch^2 u \, du &= -\coth u + C \\
\int \sech u \tanh u \, du &= -\sech u + C \\
\int \csch u \coth u \, du &= -\csch u + C
\end{align*}
\]

Proof

\[
\begin{align*}
\frac{d}{dx} \sinh x &= \frac{d}{dx} \left[ \frac{e^x - e^{-x}}{2} \right] \\
&= \frac{e^x + e^{-x}}{2} = \cosh x \\
\frac{d}{dx} \tanh x &= \frac{d}{dx} \left[ \frac{\sinh x}{\cosh x} \right] \\
&= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} \\
&= \frac{1}{\cosh^2 x} = \sech^2 x
\end{align*}
\]

In Exercises 98 and 102, you are asked to prove some of the other differentiation rules.
EXAMPLE 1  Differentiation of Hyperbolic Functions

a. \( \frac{d}{dx} \sinh(x^2 - 3) = 2x \cosh(x^2 - 3) \)

b. \( \frac{d}{dx} \ln(\cosh x) = \frac{\sinh x}{\cosh x} = \tanh x \)

c. \( \frac{d}{dx} [x \sinh x - \cosh x] = x \cosh x + \sinh x - \sinh x = x \cosh x \)

EXAMPLE 2  Finding Relative Extrema

Find the relative extrema of \( f(x) = (x - 1) \cosh x - \sinh x \).

Solution  Begin by setting the first derivative of \( f \) equal to 0.

\[
    f'(x) = (x - 1) \sinh x + \cosh x - \cosh x = 0
\]

So, the critical numbers are \( x = 1 \) and \( x = 0 \). Using the Second Derivative Test, you can verify that the point \((0, -1)\) yields a relative maximum and the point \((1, -\sinh 1)\) yields a relative minimum, as shown in Figure 5.38. Try using a graphing utility to confirm this result. If your graphing utility does not have hyperbolic functions, you can use exponential functions as follows.

\[
    f(x) = (x - 1) \left( \frac{1}{2} (e^x + e^{-x}) - \frac{1}{2} (e^x - e^{-x}) \right)
    = \frac{1}{2} (xe^x + xe^{-x} - e^x - e^{-x})
    = \frac{1}{2} (xe^x + xe^{-x} - 2e^x)
\]

When a uniform flexible cable, such as a telephone wire, is suspended from two points, it takes the shape of a catenary, as discussed in Example 3.

EXAMPLE 3  Hanging Power Cables

Power cables are suspended between two towers, forming the catenary shown in Figure 5.39. The equation for this catenary is

\[
    y = a \cosh \frac{x}{a}
\]

The distance between the two towers is \( 2b \). Find the slope of the catenary at the point where the cable meets the right-hand tower.

Solution  Differentiating produces

\[
    y' = \frac{1}{a} \sinh \frac{x}{a} = \sinh \frac{x}{a}
\]

At the point \((b, a \cosh(b/a))\), the slope (from the left) is given by \( m = \frac{b}{a} \).

FOR FURTHER INFORMATION  In Example 3, the cable is a catenary between two supports at the same height. To learn about the shape of a cable hanging between supports of different heights, see the article “Reexamining the Catenary” by Paul Cella in The College Mathematics Journal. To view this article, go to the website www.matharticles.com.
EXAMPLE 4  Integrating a Hyperbolic Function

Find \( \int \cosh 2x \sinh^2 2x \, dx \).

Solution

\[
\int \cosh 2x \sinh^2 2x \, dx = \frac{1}{2} \int (\sinh 2x)^2 (2 \cosh 2x) \, dx = \frac{1}{2} \left( \frac{(\sinh 2x)^3}{3} \right) + C = \frac{\sinh^2 2x}{6} + C.
\]

Inverse Hyperbolic Functions

Unlike trigonometric functions, hyperbolic functions are not periodic. In fact, by looking back at Figure 5.37, you can see that four of the six hyperbolic functions are actually one-to-one (the hyperbolic sine, tangent, cosecant, and cotangent). So, you can apply Theorem 5.7 to conclude that these four functions have inverse functions. The other two (the hyperbolic cosine and secant) are one-to-one if their domains are restricted to the positive real numbers, and for this restricted domain they also have inverse functions. Because the hyperbolic functions are defined in terms of exponential functions, it is not surprising to find that the inverse hyperbolic functions can be written in terms of logarithmic functions, as shown in Theorem 5.19.

**THEOREM 5.19  Inverse Hyperbolic Functions**

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) )</td>
<td>( (-\infty, \infty) )</td>
</tr>
<tr>
<td>( \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) )</td>
<td>( [1, \infty) )</td>
</tr>
<tr>
<td>( \tanh^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x} )</td>
<td>( (-1, 1) )</td>
</tr>
<tr>
<td>( \coth^{-1} x = \frac{1}{2} \ln \frac{x + 1}{x - 1} )</td>
<td>( (-\infty, -1) \cup (1, \infty) )</td>
</tr>
<tr>
<td>( \sech^{-1} x = \ln \frac{1 + \sqrt{1 - x^2}}{x} )</td>
<td>( (0, 1] )</td>
</tr>
<tr>
<td>( \csch^{-1} x = \ln \left( \frac{1}{x} + \frac{\sqrt{1 + x^2}}{</td>
<td>x</td>
</tr>
</tbody>
</table>

**Proof**  The proof of this theorem is a straightforward application of the properties of the exponential and logarithmic functions. For example, if

\[
f(x) = \sinh x = \frac{e^x - e^{-x}}{2}
\]

and

\[
g(x) = \ln(x + \sqrt{x^2 + 1})
\]

you can show that \( f(g(x)) = x \) and \( g(f(x)) = x \), which implies that \( g \) is the inverse function of \( f \).
The graphs of the inverse hyperbolic functions are shown in Figure 5.41. The inverse hyperbolic secant can be used to define a curve called a *tractrix* or *pursuit curve*, as discussed in Example 5.
EXAMPLE 5  A Tractrix

A person is holding a rope that is tied to a boat, as shown in Figure 5.42. As the person walks along the dock, the boat travels along a tractrix, given by the equation

\[ y = a \sech^{-1} \left( \frac{x}{a} \right) - \sqrt{a^2 - x^2} \]

where \( a \) is the length of the rope. If \( a = 20 \) feet, find the distance the person must walk to bring the boat 5 feet from the dock.

Solution  In Figure 5.42, notice that the distance the person has walked is given by

\[ y_1 = y + \sqrt{20^2 - x^2} = \left( 20 \sech^{-1} \left( \frac{x}{20} \right) - \sqrt{20^2 - x^2} \right) + \sqrt{20^2 - x^2} \]

\[ = 20 \sech^{-1} \left( \frac{x}{20} \right) \]

When \( x = 5 \), this distance is

\[ y_1 = 20 \sech^{-1} \left( \frac{5}{20} \right) = 20 \ln \left[ 1 + \sqrt{1 - \left( \frac{1}{4} \right)^2} \right] \]

\[ = 20 \ln \left( 4 + \sqrt{15} \right) \]

\[ \approx 41.27 \text{ feet.} \]

Differentiation and Integration of Inverse Hyperbolic Functions

The derivatives of the inverse hyperbolic functions, which resemble the derivatives of the inverse trigonometric functions, are listed in Theorem 5.20 with the corresponding integration formulas (in logarithmic form). You can verify each of these formulas by applying the logarithmic definitions of the inverse hyperbolic functions. (See Exercises 99–101.)

**THEOREM 5.20  Differentiation and Integration Involving Inverse Hyperbolic Functions**

Let \( u \) be a differentiable function of \( x \).

\[
\frac{d}{dx} \left[ \sinh^{-1} u \right] = \frac{u'}{\sqrt{u^2 + 1}} \quad \quad \frac{d}{dx} \left[ \cosh^{-1} u \right] = \frac{u'}{\sqrt{u^2 - 1}} \\
\frac{d}{dx} \left[ \tanh^{-1} u \right] = \frac{u'}{1 - u^2} \quad \quad \frac{d}{dx} \left[ \coth^{-1} u \right] = \frac{u'}{1 - u^2} \\
\frac{d}{dx} \left[ \sech^{-1} u \right] = \frac{-u'}{u \sqrt{1 - u^2}} \quad \quad \frac{d}{dx} \left[ \csch^{-1} u \right] = \frac{-u'}{|u| \sqrt{1 + u^2}} \\
\int \frac{du}{\sqrt{u^2 + a^2}} = \ln \left( u + \sqrt{u^2 + a^2} \right) + C \\
\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{a + u}{a - u} \right| + C \\
\int \frac{du}{u \sqrt{a^2 - u^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - u^2}}{|u|} \right| + C
EXAMPLE 6  More About a Tractrix

For the tractrix given in Example 5, show that the boat is always pointing toward the person.

Solution  For a point \((x, y)\) on a tractrix, the slope of the graph gives the direction of the boat, as shown in Figure 5.42.

\[
y' = \frac{d}{dx} \left[ 20 \text{sech}^{-1} \frac{x}{20} - \sqrt{20^2 - x^2} \right]
\]
\[
= -20 \left( \frac{1}{20} \right) \left[ \frac{1}{(x/20) \sqrt{1 - (x/20)^2}} \right] - \left( \frac{1}{2} \right) \left( \frac{-2x}{\sqrt{20^2 - x^2}} \right)
\]
\[
= -\frac{20^2}{x \sqrt{20^2 - x^2}} + \frac{x}{\sqrt{20^2 - x^2}}
\]
\[
= -\frac{\sqrt{20^2 - x^2}}{x}
\]

However, from Figure 5.42, you can see that the slope of the line segment connecting the point \((0, y)\) with the point \((x, y)\) is also

\[
m = -\frac{\sqrt{20^2 - x^2}}{x}.
\]

So, the boat is always pointing toward the person. (It is because of this property that a tractrix is called a \textit{pursuit curve}.)

EXAMPLE 7  Integration Using Inverse Hyperbolic Functions

Find \(\int \frac{dx}{x \sqrt{4 - 9x^2}}\).

Solution  Let \(a = 2\) and \(u = 3x\).

\[
\int \frac{dx}{x \sqrt{4 - 9x^2}} = \int \frac{3 \, dx}{(3x) \sqrt{4 - 9x^2}} = \int \frac{du}{u \sqrt{u^2 - a^2}} = -\frac{1}{a} \ln \frac{u + \sqrt{u^2 - a^2}}{a} + C
\]

EXAMPLE 8  Integration Using Inverse Hyperbolic Functions

Find \(\int \frac{dx}{5 - 4x^2}\).

Solution  Let \(a = \sqrt{5}\) and \(u = 2x\).

\[
\int \frac{dx}{5 - 4x^2} = \frac{1}{2} \int \frac{2 \, dx}{(\sqrt{5})^2 - (2x)^2} = \frac{1}{2} \ln \left[ \frac{\sqrt{5} + 2x}{\sqrt{5} - 2x} \right] + C
\]
\[
= \frac{1}{4\sqrt{5}} \ln \left[ \frac{\sqrt{5} + 2x}{\sqrt{5} - 2x} \right] + C
\]
In Exercises 1–6, evaluate the function. If the value is not a rational number, give the answer to three-decimal-place accuracy.

1. (a) \( \sinh 3 \)
   (b) \( \tanh(-2) \)
2. (a) \( \cosh 0 \)
   (b) \( \text{sech} 1 \)
3. (a) \( \cosh(\ln 2) \)
   (b) \( \coth(\ln 5) \)
4. (a) \( \sinh^{-1} 0 \)
   (b) \( \tanh^{-1} 0 \)
5. (a) \( \cosh^{-1} \frac{5}{3} \)
   (b) \( \text{sech}^{-1} \frac{3}{2} \)
6. (a) \( \cosh^{-1} 2 \)
   (b) \( \coth^{-1} 2 \)

In Exercises 7–12, verify the identity.

7. \( \tanh^2 x + \text{sech}^2 x = 1 \)
8. \( \cosh^2 x = \frac{1 + \cosh 2x}{2} \)
9. \( \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \)
10. \( \sinh 2x = 2 \sinh x \cosh x \)
11. \( \sinh 3x = 3 \sinh x + 4 \sinh^3 x \)
12. \( \cosh x + \cosh y = 2 \cosh \frac{x + y}{2} \cosh \frac{x - y}{2} \)

In Exercises 13 and 14, use the value of the given hyperbolic function to find the values of the other hyperbolic functions at \( x \).

13. \( \sinh x = \frac{3}{2} \)
14. \( \tanh x = \frac{1}{2} \)

In Exercises 15–24, find the derivative of the function.

15. \( y = \text{sech}(x + 1) \)
16. \( y = \coth 3x \)
17. \( f(x) = \ln(\sinh x) \)
18. \( g(x) = \ln(\cosh x) \)
19. \( y = \ln(\tanh \frac{x}{2}) \)
20. \( y = x \cosh x - \sinh x \)
21. \( h(x) = \frac{1}{4} \sinh 2x - \frac{x}{2} \)
22. \( h(t) = t - \coth t \)
23. \( f(t) = \arctan(\sinh t) \)
24. \( g(x) = \text{sech}^2 3x \)

In Exercises 25–28, find an equation of the tangent line to the graph of the function at the given point.

25. \( y = \sinh(1 - x^2) \)
26. \( y = x \cosh x \)

27. \( y = (\cosh x - \sinh x)^2 \)
28. \( y = e^{\sinh x} \)

In Exercises 29–32, find any relative extrema of the function. Use a graphing utility to confirm your result.

29. \( f(x) = \sin x \sinh x - \cos x \cosh x, \quad -4 \leq x \leq 4 \)
30. \( f(x) = x \sinh(x - 1) - \cosh(x - 1) \)
31. \( g(x) = x \sech x \)
32. \( h(x) = 2 \tanh x - x \)

In Exercises 33 and 34, show that the function satisfies the differential equation.

\begin{align*}
\text{Function} & & \text{Differential Equation} \\
33. y = a \sinh x & & y'' - y' = 0 \\
34. y = a \cosh x & & y'' - y = 0
\end{align*}

Linear and Quadratic Approximations In Exercises 35 and 36, use a computer algebra system to find the linear approximation \( P_1(x) = f(a) + f'(a)(x - a) \) and the quadratic approximation \( P_2(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 \) of the function \( f \) at \( x = a \). Use a graphing utility to graph the function and its linear and quadratic approximations.

35. \( f(x) = \tanh x, \quad a = 0 \)
36. \( f(x) = \cosh x, \quad a = 0 \)

Catenary In Exercises 37 and 38, a model for a power cable suspended between two towers is given. (a) Graph the model, (b) find the heights of the cable at the towers and at the midpoint between the towers, and (c) find the slope of the model at the point where the cable meets the right-hand tower.

37. \( y = 10 + 15 \cosh \frac{x}{15}, \quad -15 \leq x \leq 15 \)
38. \( y = 18 + 25 \cosh \frac{x}{25}, \quad -25 \leq x \leq 25 \)

In Exercises 39–50, find the integral.

39. \( \int \sinh(1 - 2x) \, dx \)
40. \( \int \frac{\cosh \sqrt{x}}{\sqrt{x}} \, dx \)
41. \( \int \cosh^2(x - 1) \sinh(x - 1) \, dx \)
42. \( \int \frac{\sinh x}{1 + \sinh^2 x} \, dx \)
In Exercises 67–72, find the limit.

43. \( \int \frac{\cosh x}{\sinh x} \, dx \)
44. \( \int \text{sech}^2(2x - 1) \, dx \)
45. \( \int x \cosh \frac{x^2}{2} \, dx \)
46. \( \int \text{sech}^3 x \tanh x \, dx \)
47. \( \int \frac{\cosh(1/x)}{\sinh(1/x)} \, dx \)
48. \( \int \frac{\cosh x}{\sqrt{9 - \sinh^2 x}} \, dx \)
49. \( \int \frac{x}{x^2 + 1} \, dx \)
50. \( \int \frac{2}{x \sqrt{1 + 4x^2}} \, dx \)

In Exercises 51–56, evaluate the integral.

51. \( \int_0^{\ln 2} \tan x \, dx \)
52. \( \int_0^4 \cosh^2 x \, dx \)
53. \( \int_{\pi/4}^0 \frac{1}{25 - x^2} \, dx \)
54. \( \int_0^{\pi/2} \frac{1}{\sqrt{25 - x^2}} \, dx \)
55. \( \int_0^2 \frac{2}{\sqrt{1 - 4x^2}} \, dx \)
56. \( \int_0^4 2e^{-x} \cosh x \, dx \)

In Exercises 57–64, find the derivative of the function.

57. \( y = \cosh^{-1}(3x) \)
58. \( y = \tanh^{-1} \frac{x}{2} \)
59. \( y = \sinh^{-1}(\tan x) \)
60. \( y = \text{sech}^{-1} (\cos 2x), \quad 0 < x < \pi/4 \)
61. \( y = \tanh^{-1} (\sin 2x) \)
62. \( y = \text{sech}^{-1}(x) \)
63. \( y = 2\sqrt{x} \sinh^{-1}(2x) - \sqrt{1 + 4x^2} \)
64. \( y = x \tanh^{-1} x + \ln \sqrt{1 - x^2} \)

Writing About Concepts

65. Discuss several ways in which the hyperbolic functions are similar to the trigonometric functions.
66. Sketch the graph of each hyperbolic function. Then identify the domain and range of each function.

Limits

In Exercises 67–72, find the limit.

67. \( \lim_{x \to \infty} \sinh x \)
68. \( \lim_{x \to \infty} \tanh x \)
69. \( \lim_{x \to \infty} \text{sech} x \)
70. \( \lim_{x \to \infty} \cosh x \)
71. \( \lim_{x \to 0} \frac{\sinh x}{x} \)
72. \( \lim_{x \to 0} \cosh x \)

In Exercises 73–80, find the indefinite integral using the formulas of Theorem 5.20.

73. \( \int \frac{1}{\sqrt{1 + e^{2x}}} \, dx \)
74. \( \int \frac{x}{9 - x^4} \, dx \)
75. \( \int \frac{1}{\sqrt{x^2 + x}} \, dx \)
76. \( \int \frac{\sqrt{x}}{4 + x^2} \, dx \)
77. \( \int \frac{-1}{4x - x^2} \, dx \)
78. \( \int \frac{dx}{(x + 2)\sqrt{x^2 + 4x + 8}} \)
79. \( \int \frac{1}{1 - 4x^2} \, dx \)
80. \( \int \frac{dx}{x(1 + \sqrt{2x^2 + 4x + 8})} \)

In Exercises 81–84, solve the differential equation.

81. \( \frac{dy}{dx} = \frac{1}{\sqrt{80 + 8x - 16x^2}} \)
82. \( \frac{dy}{dx} = \frac{1}{(x - 1)\sqrt{4x^2 + 8x - 1}} \)
83. \( \frac{dy}{dx} = \frac{x^3 - 21x}{5 + 4x - x^2} \)
84. \( \frac{dy}{dx} = \frac{1 - 2x}{4x - x^2} \)

Area

In Exercises 85–88, find the area of the region.

85. \( y = \frac{x}{2} \)
86. \( y = \tanh 2x \)
87. \( y = \frac{5x}{\sqrt{x^2 + 4}} \)
88. \( y = \frac{6}{\sqrt{x^2 - 4}} \)

In Exercises 89 and 90, evaluate the integral in terms of (a) natural logarithms and (b) inverse hyperbolic functions.

89. \( \int_0^1 \frac{\sqrt{x}}{\sqrt{x^2 + 1}} \, dx \)
90. \( \int_{-1/2}^{1/2} \frac{dx}{1 - x^2} \)

Chemical Reactions

Chemicals A and B combine in a 3-to-1 ratio to form a compound. The amount of compound x being produced at any time t is proportional to the unchanged amounts of A and B remaining in the solution. So, if 3 kilograms of A is mixed with 2 kilograms of B, you have

\[
\frac{dx}{dt} = k\left(3 - \frac{3x}{4}\right)(2 - \frac{x}{4}) = \frac{3k}{16}(x^2 - 12x + 32).
\]

One kilogram of the compound is formed after 10 minutes. Find the amount formed after 20 minutes by solving the equation

\[
\int \frac{3k}{16} \, dt = \int \frac{dx}{x^2 - 12x + 32}.
\]
92. **Vertical Motion** An object is dropped from a height of 400 feet.

(a) Find the velocity of the object as a function of time (neglect air resistance on the object).

(b) Use the result in part (a) to find the position function.

(c) If the air resistance is proportional to the square of the velocity, then \( \frac{dv}{dt} = -32 + kv^2 \), where \(-32\) feet per second per second is the acceleration due to gravity and \( k \) is a constant. Show that the velocity \( v \) as a function of time is \[
v(t) = -\sqrt{\frac{32}{k}} \tanh\left(\sqrt{\frac{32}{k}} t\right)
\]
by performing the following integration and simplifying the result.

\[
\int \frac{dv}{32 - kv^2} = -\int dt
\]

(d) Use the result in part (c) to find \( \lim_{t \to \infty} v(t) \) and give its interpretation.

(e) Integrate the velocity function in part (c) and find the position \( x \) of the object as a function of \( t \). Use a graphing utility to graph the position function when \( k = 0.01 \) and the position function in part (b) in the same viewing window. Estimate the additional time required for the object to reach ground level when air resistance is not neglected.

(f) Give a written description of what you believe would happen if \( k \) were increased. Then test your assertion with a particular value of \( k \).

**Tractrix** In Exercises 93 and 94, use the equation of the tractrix

\[
y = a \sech^{-1} \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0.
\]

93. Find \( dy/dx \).

94. Let \( L \) be the tangent line to the tractrix at the point \( P \). If \( L \) intersects the \( y \)-axis at the point \( Q \), show that the distance between \( P \) and \( Q \) is \( a \).

95. Prove that \( \tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right), \quad -1 < x < 1 \).

96. Show that \( \arctan(\sinh x) = \arcsin(\tanh x) \).

97. Let \( x > 0 \) and \( b > 0 \). Show that \[
\int_{-b}^{b} e^{x^2} dx = \frac{2 \sinh bx}{x}.
\]

In Exercises 98–102, verify the differentiation formula.

98. \( \frac{d}{dx}[\cosh x] = \sinh x \)

99. \( \frac{d}{dx}[\sech^{-1} x] = \frac{-1}{x \sqrt{1 - x^2}} \)

100. \( \frac{d}{dx}[\cosh^{-1} x] = \frac{1}{\sqrt{x^2 - 1}} \)

101. \( \frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{x^2 + 1}} \)

102. \( \frac{d}{dx}[\sech x] = -\sech x \tanh x \)

**Putnam Exam Challenge**

103. From the vertex \((0, c)\) of the catenary \( y = c \cosh \left( x/c \right) \) a line \( L \) is drawn perpendicular to the tangent to the catenary at a point \( P \). Prove that the length of \( L \) intercepted by the axes is equal to the ordinate \( y \) of the point \( P \).

104. Prove or disprove that there is at least one straight line normal to the graph of \( y = \cosh x \) at a point \((a, \cosh a)\) and also normal to the graph of \( y = \sinh x \) at a point \((c, \sinh c)\).

[At a point on a graph, the normal line is the perpendicular to the tangent at that point. Also, \( \cosh x = (e^x + e^{-x})/2 \) and \( \sinh x = (e^x - e^{-x})/2 \).]

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**Section Project: St. Louis Arch**

The Gateway Arch in St. Louis, Missouri was constructed using the hyperbolic cosine function. The equation used for construction was \( y = 693.8597 - 68.7672 \cosh 0.0100333x, \)

\(-299.2239 \leq x \leq 299.2239 \)

where \( x \) and \( y \) are measured in feet. Cross sections of the arch are equilateral triangles, and \((x, y)\) traces the path of the centers of mass of the cross-sectional triangles. For each value of \( x \), the area of the cross-sectional triangle is \( A = 125.1406 \cosh 0.0100333x \).

(Source: Owner’s Manual for the Gateway Arch, Saint Louis, MO, by William Thayer)

(a) How high above the ground is the center of the highest triangle? (At ground level, \( y = 0 \).)

(b) What is the height of the arch? (Hint: For an equilateral triangle, \( A = \sqrt{3}c^2 \), where \( c \) is one-half the base of the triangle, and the center of mass of the triangle is located at two-thirds the height of the triangle.)

(c) How wide is the arch at ground level?