

**Section 9.2 Series and Convergence**

- Understand the definition of a convergent infinite series.
- Use properties of infinite geometric series.
- Use the \( n \)-th Term Test for Divergence of an infinite series.

**Infinite Series**

One important application of infinite sequences is in representing “infinite summations.” Informally, if \( \{a_n\} \) is an infinite sequence, then

\[
\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots
\]

is an **infinite series** (or simply a **series**). The numbers \( a_1, a_2, a_3, \) are the **terms** of the series. For some series it is convenient to begin the index at \( n = 0 \) (or some other integer). As a typesetting convention, it is common to represent an infinite series as \( \sum a_n \). In such cases, the starting value for the index must be taken from the context of the statement.

To find the sum of an infinite series, consider the following sequence of partial sums.

\[
S_1 = a_1
\]

\[
S_2 = a_1 + a_2
\]

\[
S_3 = a_1 + a_2 + a_3
\]

\[
\vdots
\]

\[
S_n = a_1 + a_2 + a_3 + \ldots + a_n
\]

If this sequence of partial sums converges, the series is said to converge and has the sum indicated in the following definition.

**Definitions of Convergent and Divergent Series**

For the infinite series \( \sum_{n=1}^{\infty} a_n \), the **\( n \)-th partial sum** is given by

\[
S_n = a_1 + a_2 + \ldots + a_n.
\]

If the sequence of partial sums \( \{S_n\} \) converges to \( S \), then the series \( \sum_{n=1}^{\infty} a_n \) **converges**. The limit \( S \) is called the **sum of the series**.

\[
S = a_1 + a_2 + \ldots + a_n + \ldots
\]

If \( \{S_n\} \) diverges, then the series **diverges**.

**STUDY TIP:** As you study this chapter, you will see that there are two basic questions involving infinite series. Does a series converge or does it diverge? If a series converges, what is its sum? These questions are not always easy to answer, especially the second one.

**STUDY TIP:**

**Finding the Sum of an Infinite Series**

Find the sum of each infinite series. Explain your reasoning.

| a. \( 0.1 + 0.01 + 0.001 + 0.0001 + \ldots \) | b. \( \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \ldots \) |
| c. \( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \) | d. \( \frac{15}{10000} + \frac{15}{100000} + \frac{15}{1000000} + \ldots \) |
SECTION 9.2 Series and Convergence

EXAMPLE 1 Convergent and Divergent Series

a. The series
\[ \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \]
has the following partial sums.

\[ S_1 = \frac{1}{2} \]
\[ S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \]
\[ S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \]
\[ \vdots \]
\[ S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n} \]

Because
\[ \lim_{n \to \infty} \frac{2^n - 1}{2^n} = 1 \]
it follows that the series converges and its sum is 1.

b. The nth partial sum of the series
\[ \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \]
is given by
\[ S_n = 1 - \frac{1}{n+1} \]

Because the limit of \( S_n \) is 1, the series converges and its sum is 1.

c. The series
\[ \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \cdots \]
diverges because \( S_n = n \) and the sequence of partial sums diverges.

The series in Example 1(b) is a telescoping series of the form
\[ (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots \]

FOR FURTHER INFORMATION To learn more about the partial sums of infinite series, see the article “Six Ways to Sum a Series” by Dan Kalman in The College Mathematics Journal. To view this article, go to the website www.matharticles.com.
EXAMPLE 2 Writing a Series in Telescoping Form

Find the sum of the series ∑_{n=1}^{∞} \frac{2}{4n^2 - 1}.

Solution
Using partial fractions, you can write

\[ a_n = \frac{2}{4n^2 - 1} = \frac{2}{(2n - 1)(2n + 1)} = \frac{1}{2n - 1} - \frac{1}{2n + 1}. \]

From this telescoping form, you can see that the nth partial sum is

\[ S_n = \left( \frac{1}{1} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \cdots + \left( \frac{1}{2n - 1} - \frac{1}{2n + 1} \right) = 1 - \frac{1}{2n + 1}. \]

So, the series converges and its sum is 1. That is,

\[ \sum_{n=1}^{∞} \frac{2}{4n^2 - 1} = \lim_{n→∞} S_n = \lim_{n→∞} \left( 1 - \frac{1}{2n + 1} \right) = 1. \]

Geometric Series
The series given in Example 1(a) is a geometric series. In general, the series given by

\[ \sum_{n=0}^{∞} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, \quad a ≠ 0 \]

is a geometric series with ratio r.

THEOREM 9.6 Convergence of a Geometric Series
A geometric series with ratio r diverges if |r| ≥ 1. If 0 < |r| < 1, then the series converges to the sum

\[ \sum_{n=0}^{∞} ar^n = \frac{a}{1 - r}, \quad 0 < |r| < 1. \]

Proof It is easy to see that the series diverges if r = ±1. If r ≠ ±1, then \( S_n = a + ar + ar^2 + \cdots + ar^{n-1} \). Multiplication by r yields

\[ rS_n = ar + ar^2 + ar^3 + \cdots + ar^n. \]

Subtracting the second equation from the first produces \( S_n - rS_n = a - ar^n \). Therefore, \( S_n(1 - r) = a(1 - r^n) \), and the nth partial sum is

\[ S_n = \frac{a}{1 - r}(1 - r^n). \]

If 0 < |r| < 1, it follows that \( r^n → 0 \) as \( n → ∞ \), and you obtain

\[ \lim_{n→∞} S_n = \lim_{n→∞} \left[ \frac{a}{1 - r} (1 - r^n) \right] = \frac{a}{1 - r} \left[ \lim_{n→∞} (1 - r^n) \right] = \frac{a}{1 - r}, \]

which means that the series converges and its sum is \( a/(1 - r) \). It is left to you to show that the series diverges if |r| > 1.
EXAMPLE 3  Convergent and Divergent Geometric Series

a. The geometric series
\[ \sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n = 3 \left( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \right) \]

has a ratio of \( r = \frac{1}{2} \) with \( a = 3 \). Because \( 0 < |r| < 1 \), the series converges and its sum is
\[ S = \frac{a}{1 - r} = \frac{3}{1 - (1/2)} = 6. \]

b. The geometric series
\[ \sum_{n=0}^{\infty} \left( \frac{3}{2} \right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots \]

has a ratio of \( r = \frac{3}{2} \). Because \( |r| \geq 1 \), the series diverges.

The formula for the sum of a geometric series can be used to write a repeating decimal as the ratio of two integers, as demonstrated in the next example.

EXAMPLE 4  A Geometric Series for a Repeating Decimal

Use a geometric series to write \( 0.0\overline{8} \) as the ratio of two integers.

Solution  For the repeating decimal \( 0.0\overline{8} \), you can write
\[ 0.080808 \cdots = \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} + \cdots \]

\[ = \sum_{n=0}^{\infty} \left( \frac{8}{10^2} \right) \left( \frac{1}{10^2} \right)^n \]

For this series, you have \( a = 8/10^2 \) and \( r = 1/10^2 \). So,
\[ 0.080808 \cdots = \frac{a}{1 - r} = \frac{8/10^2}{1 - (1/10^2)} = \frac{8}{99}. \]

Try dividing 8 by 99 on a calculator to see that it produces \( 0.0\overline{8} \).

The convergence of a series is not affected by removal of a finite number of terms from the beginning of the series. For instance, the geometric series
\[ \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \quad \text{and} \quad \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \]

both converge. Furthermore, because the sum of the second series is \( a/(1 - r) = 2 \), you can conclude that the sum of the first series is
\[ S = 2 - \left[ \left( \frac{1}{2} \right)^0 + \left( \frac{1}{2} \right)^1 + \left( \frac{1}{2} \right)^2 + \left( \frac{1}{2} \right)^3 \right] \]
\[ = 2 - \frac{15}{8} = \frac{1}{8} \]
CHAPTER 9  Infinite Series

The following properties are direct consequences of the corresponding properties of limits of sequences.

**THEOREM 9.7  Properties of Infinite Series**

If \( \sum a_n = A, \sum b_n = B, \) and \( c \) is a real number, then the following series converge to the indicated sums.

1. \( \sum_{n=1}^{\infty} ca_n = cA \)
2. \( \sum_{n=1}^{\infty} (a_n + b_n) = A + B \)
3. \( \sum_{n=1}^{\infty} (a_n - b_n) = A - B \)

### nth-Term Test for Divergence

The following theorem states that if a series converges, the limit of its \( n \)th term must be 0.

**THEOREM 9.8  Limit of nth Term of a Convergent Series**

If \( \sum_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0. \)

**Proof**

Assume that

\[
\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n = L.
\]

Then, because \( S_n = S_{n-1} + a_n \) and

\[
\lim_{n \to \infty} S_n = \lim_{n \to \infty} S_{n-1} = L
\]

it follows that

\[
L = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left( S_{n-1} + a_n \right)
= \lim_{n \to \infty} S_{n-1} + \lim_{n \to \infty} a_n
= L + \lim_{n \to \infty} a_n
\]

which implies that \( \{a_n\} \) converges to 0.

The contrapositive of Theorem 9.8 provides a useful test for divergence. This **nth-Term Test for Divergence** states that if the limit of the \( n \)th term of a series does not converge to 0, the series must diverge.

**THEOREM 9.9  nth-Term Test for Divergence**

If \( \lim_{n \to \infty} a_n \neq 0, \) then \( \sum_{n=1}^{\infty} a_n \) diverges.
EXAMPLE 5  Using the $n$th-Term Test for Divergence

a. For the series $\sum_{n=0}^{\infty} 2^n$, you have
$$\lim_{n\to\infty} 2^n = \infty.$$ 
So, the limit of the $n$th term is not 0, and the series diverges.

b. For the series $\sum_{n=1}^{\infty} \frac{n!}{2n! + 1}$, you have
$$\lim_{n\to\infty} \frac{n!}{2n! + 1} = \frac{1}{2}.$$ 
So, the limit of the $n$th term is not 0, and the series diverges.

c. For the series $\sum_{n=1}^{\infty} \frac{1}{n}$, you have
$$\lim_{n\to\infty} \frac{1}{n} = 0.$$ 
Because the limit of the $n$th term is 0, the $n$th-Term Test for Divergence does not apply and you can draw no conclusions about convergence or divergence. (In the next section, you will see that this particular series diverges.)

EXAMPLE 6  Bouncing Ball Problem

A ball is dropped from a height of 6 feet and begins bouncing, as shown in Figure 9.7. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance traveled by the ball.

Solution  When the ball hits the ground for the first time, it has traveled a distance of $D_1 = 6$ feet. For subsequent bounces, let $D_i$ be the distance traveled up and down. For example, $D_2$ and $D_3$ are as follows.

$$D_2 = 6\left(\frac{1}{4}\right) + 6\left(\frac{1}{4}\right) = 12\left(\frac{1}{4}\right)$$ 
Up  Down

$$D_3 = 6\left(\frac{1}{4}\right)^2 + 6\left(\frac{1}{4}\right)^2 = 12\left(\frac{1}{4}\right)^2$$ 
Up  Down

By continuing this process, it can be determined that the total vertical distance is

$$D = 6 + 12\left(\frac{1}{4}\right) + 12\left(\frac{1}{4}\right)^2 + 12\left(\frac{1}{4}\right)^3 + \cdots$$

$$= 6 + 12 \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$$

$$= 6 + 9 \left(1 - \frac{1}{2}\right)$$

$$= 6 + 9(4)$$

$$= 42 \text{ feet.}$$
In Exercises 1–6, find the first five terms of the sequence of partial sums.

1. \(1 + \frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + \cdots\)
2. \(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \cdots\)
3. \(\frac{2}{3} - \frac{1}{6} - \frac{1}{11} - \frac{1}{18} - \cdots\)
4. \(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\)
5. \(\sum_{n=1}^{\infty} \frac{3}{n^2 - 1}\)
6. \(\sum_{n=1}^{\infty} (-1)^{n+1}\)

In Exercises 7–16, verify that the infinite series diverges.

7. \(\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n\)
8. \(\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n\)
9. \(\sum_{n=0}^{\infty} 1000(1.055)^n\)
10. \(\sum_{n=0}^{\infty} 2(-1.03)^n\)
11. \(\sum_{n=1}^{\infty} \frac{n}{n+1}\)
12. \(\sum_{n=1}^{\infty} \frac{n}{2n+3}\)
13. \(\sum_{n=1}^{\infty} \frac{n^2}{n+1}\)
14. \(\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 1}}\)
15. \(\sum_{n=1}^{\infty} \frac{2^n + 1}{2n+1}\)
16. \(\sum_{n=1}^{\infty} \frac{n!}{2^n}\)

In Exercises 17–22, match the series with the graph of its sequence of partial sums. (The graphs are labeled (a), (b), (c), (d), (e), and (f).) Use the graph to estimate the sum of the series. Confirm your answer analytically.

In Exercises 23–28, verify that the infinite series converges.

23. \(\sum_{n=1}^{\infty} \frac{1}{n(n+1)}\) (Use partial fractions.)
24. \(\sum_{n=1}^{\infty} \frac{1}{n(n+2)}\) (Use partial fractions.)
25. \(\sum_{n=1}^{\infty} \frac{3^n}{4^n}\)
26. \(\sum_{n=1}^{\infty} 2 \left(\frac{1}{2}\right)^n\)
27. \(\sum_{n=0}^{\infty} (0.9)^n = 1 + 0.9 + 0.81 + 0.729 + \cdots\)
28. \(\sum_{n=0}^{\infty} (-0.6)^n = 1 - 0.6 + 0.36 - 0.216 + \cdots\)

**Numerical, Graphical, and Analytic Analysis**

In Exercises 29–34, (a) find the sum of the series, (b) use a graphing utility to find the indicated partial sum \(S_n\) and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum, and (d) explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.

<table>
<thead>
<tr>
<th>(n)</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_n)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

29. \(\sum_{n=1}^{\infty} \frac{6}{n(n+3)}\)
30. \(\sum_{n=1}^{\infty} \frac{4}{n(n+4)}\)
31. \(\sum_{n=1}^{\infty} 2(0.9)^{n-1}\)
32. \(\sum_{n=1}^{\infty} 3(0.85)^{n-1}\)
33. \(\sum_{n=1}^{\infty} 10(0.25)^{n-1}\)
34. \(\sum_{n=1}^{\infty} 5 \left(\frac{1}{3}\right)^{n-1}\)

In Exercises 35–50, find the sum of the convergent series.

35. \(\sum_{n=1}^{\infty} \frac{1}{n^2 - 1}\)
36. \(\sum_{n=1}^{\infty} \frac{4}{n(n+2)}\)
In Exercises 51–56, (a) write the repeating decimal as a geometric series and (b) write its sum as the ratio of two integers.

51. 0.3
52. 0.37
53. 0.87
54. 0.75
55. 0.075
56. 0.275

In Exercises 57–72, determine the convergence or divergence of the series.

57. \( \sum_{n=1}^{\infty} \frac{n + 10}{10n + 1} \)
58. \( \sum_{n=1}^{\infty} \frac{n + 1}{2n - 1} \)
59. \( \sum_{n=1}^{\infty} \left( \frac{1}{n - 1} + \frac{1}{n + 2} \right) \)
60. \( \sum_{n=1}^{\infty} \frac{1}{n(n + 3)} \)
61. \( \sum_{n=1}^{\infty} \frac{3n - 1}{2n + 1} \)
62. \( \sum_{n=1}^{\infty} \frac{3^n}{n^2} \)
63. \( \sum_{n=0}^{\infty} \frac{4}{3^n} \)
64. \( \sum_{n=0}^{\infty} \frac{1}{n!(n+2)} \)
65. \( \sum_{n=0}^{\infty} (1.075)^n \)
66. \( \sum_{n=0}^{\infty} \frac{2^n}{100} \)
67. \( \sum_{n=1}^{\infty} \frac{n^2}{n^2 + 1} \)
68. \( \sum_{n=1}^{\infty} \frac{n}{n!} \)
69. \( \sum_{n=1}^{\infty} \left( 1 + \frac{k}{n} \right)^n \)
70. \( \sum_{n=1}^{\infty} \frac{e^{-n}}{n!} \)
71. \( \sum_{n=1}^{\infty} \arctan n \)
72. \( \sum_{n=1}^{\infty} \log \left( \frac{n + 1}{n} \right) \)

Writing About Concepts

73. State the definitions of convergent and divergent series.
74. Describe the difference between \( \lim_{n \to \infty} a_n = 5 \) and \( \sum_{n=1}^{\infty} a_n = 5 \).
75. Define a geometric series, state when it converges, and give the formula for the sum of a convergent geometric series.

Writing About Concepts (continued)

76. State the nth-Term Test for Divergence.
77. Let \( a_n = \frac{n + 1}{n} \). Discuss the convergence of \( \{a_n\} \) and \( \sum_{n=1}^{\infty} a_n \).
78. Explain any differences among the following series.
   (a) \( \sum_{n=1}^{\infty} a_n \)  (b) \( \sum_{n=1}^{\infty} a_k \)  (c) \( \sum_{n=1}^{\infty} a_k \)

In Exercises 79–86, find all values of \( x \) for which the series converges. For these values of \( x \), write the sum of the series as a function of \( x \).

79. \( \sum_{n=0}^{\infty} x^n \)
80. \( \sum_{n=0}^{\infty} \left( \frac{3}{x} \right)^n \)
81. \( \sum_{n=0}^{\infty} \left( x - 1 \right)^n \)
82. \( \sum_{n=0}^{\infty} \left( \frac{3}{2} - \frac{3}{4} \right)^n \)
83. \( \sum_{n=0}^{\infty} \left( -1 \right)^n x^n \)
84. \( \sum_{n=0}^{\infty} \left( -1 \right)^n x^{2n} \)
85. \( \sum_{n=0}^{\infty} \left( \frac{1}{x} \right)^n \)
86. \( \sum_{n=0}^{\infty} \left( \frac{1}{x^2 + 4} \right)^n \)

87. (a) You delete a finite number of terms from a divergent series.
   Will the new series still diverge? Explain your reasoning.
   (b) You add a finite number of terms to a convergent series.
   Will the new series still converge? Explain your reasoning.

88. Think About It
   Consider the formula
   \( \frac{1}{x - 1} = 1 + x + x^2 + x^3 + \cdots \).
   Given \( x = -1 \) and \( x = 2 \), can you conclude that either of the following statements is true? Explain your reasoning.
   (a) \( \frac{1}{2} = -1 + 1 - 1 + \cdots \)
   (b) \(-1 = 1 + 2 + 4 + 8 + \cdots \)

In Exercises 89 and 90, (a) find the common ratio of the geometric series, (b) write the function that gives the sum of the series, and (c) use a graphing utility to graph the function and the partial sums \( S_3 \) and \( S_6 \). What do you notice?

89. \( 1 + x + x^2 + x^3 + \cdots \)
90. \( 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots \)

In Exercises 91 and 92, use a graphing utility to graph the function. Identify the horizontal asymptote of the graph and determine its relationship to the sum of the series.

Function
| \( f(x) = 3 \left( \frac{1 - (0.5)^n}{1 - 0.5} \right) \) |
| \( \sum_{n=0}^{\infty} \left( \frac{1}{5} \right)^n \) |

Series
| \( f(x) = 2 \left( \frac{1 - (0.8)^n}{1 - 0.8} \right) \) |
| \( \sum_{n=0}^{\infty} \left( \frac{4}{5} \right)^n \) |
Writing  In Exercises 93 and 94, use a graphing utility to determine the first term that is less than 0.0001 in each of the convergent series. Note that the answers are very different. Explain how this will affect the rate at which the series converges.

93. \[
\sum_{n=1}^{\infty} \frac{1}{n(n+1)}, \quad \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n
\]

94. \[
\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} (0.01)^n
\]

Marketing  An electronic games manufacturer producing a new product estimates the annual sales to be 8000 units. Each year 10% of the units that have been sold will become inoperative. So, 8000 units will be in use after 1 year, [8000 + 0.9(8000)] units will be in use after 2 years, and so on. How many units will be in use after \(n\) years?

Depreciation  A company buys a machine for \$225,000 that depreciates at a rate of 30% per year. Find a formula for the value of the machine after \(n\) years. What is its value after 5 years?

Multiplier Effect  The annual spending by tourists in a resort city is \$100 million. Approximately 75% of that revenue is again spent in the resort city, and of that amount approximately 75% is again spent in the same city, and so on. Write the geometric series that gives the total amount of spending generated by the \$100 million and find the sum of the series.

Multiplier Effect  Repeat Exercise 97 if the percent of the revenue that is spent again in the city decreases to 60%.

Distance  A ball is dropped from a height of 16 feet. Each time it drops \(h\) feet, it rebounds 0.81\(h\) feet. Find the total distance traveled by the ball.

Time  The ball in Exercise 99 takes the following times for each fall.

\[s_1 = -16t^2 + 16, \quad s_1 = 0 \text{ if } t = 1\]
\[s_2 = -16t^2 + 16(0.81), \quad s_2 = 0 \text{ if } t = 0.9\]
\[s_3 = -16t^2 + 16(0.81)^2, \quad s_3 = 0 \text{ if } t = (0.9)^2\]
\[s_4 = -16t^2 + 16(0.81)^3, \quad s_4 = 0 \text{ if } t = (0.9)^3\]
\[\vdots\]
\[s_n = -16t^2 + 16(0.81)^{n-1}, \quad s_n = 0 \text{ if } t = (0.9)^{n-1}\]

Beginning with \(s_2\), the ball takes the same amount of time to bounce up as it does to fall, and so the total time elapsed before it comes to rest is given by

\[t = 1 + 2\sum_{n=1}^{\infty} (0.9)^n\]

Find this total time.

Probability  In Exercises 101 and 102, the random variable \(n\) represents the number of units of a product sold per day in a store. The probability distribution of \(n\) is given by \(P(n)\). Find the probability that two units are sold in a given day \([P(2)]\) and show that \(P(1) + P(2) + P(3) + \cdots = 1\).

\[P(n) = \frac{1}{2} \left( \frac{1}{2} \right)^n\]

\[P(n) = \frac{1}{3} \left( \frac{2}{3} \right)^n\]

103. Probability  A fair coin is tossed repeatedly. The probability that the first head occurs on the \(n\)th toss is given by \(P(n) = \left( \frac{1}{2} \right)^n\), where \(n \geq 1\).

(a) Show that \(\sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n = 1\).

(b) The expected number of tosses required until the first head occurs in the experiment is given by

\[\sum_{n=1}^{\infty} n \left( \frac{1}{2} \right)^n\]

Is this series geometric?

(c) Use a computer algebra system to find the sum in part (b).

104. Probability  In an experiment, three people toss a fair coin one at a time until one of them tosses a head. Determine, for each person, the probability that he or she tosses the first head. Verify that the sum of the three probabilities is 1.

105. Area  The sides of a square are 16 inches in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the triangles outside the second square are shaded (see figure). Determine the area of the shaded regions (a) if this process is continued five more times and (b) if this pattern of shading is continued infinitely.

In Exercises 107–110, use the formula for the \(n\)th partial sum of a geometric series

\[\sum_{i=1}^{n} a r^i = \frac{a(1 - r^n)}{1 - r}\]

106. Length  A right triangle \(XYZ\) is shown above where \(\overline{XY} = z\) and \(\angle X = \theta\). Line segments are continually drawn to be perpendicular to the triangle, as shown in the figure.

(a) Find the total length of the perpendicular line segments \(\overline{Y_1X_1} + \overline{X_1Y_2} + \overline{Y_2X_3} + \cdots\) in terms of \(z\) and \(\theta\).

(b) If \(z = 1\) and \(\theta = \pi/6\), find the total length of the perpendicular line segments.

107. Present Value  The winner of a $1,000,000 sweepstakes will be paid $50,000 per year for 20 years. The money earns 6% interest per year. The present value of the winnings is

\[\sum_{n=1}^{20} 50,000 \left( \frac{1}{1.06} \right)^n\]

Compute the present value and interpret its meaning.
108. Sphereflake A sphereflake shown below is a computer-generated fractal that was created by Eric Haines. The radius of the large sphere is 1. To the large sphere, nine spheres of radius $\frac{1}{2}$ are attached. To each of these, nine spheres of radius $\frac{1}{3}$ are attached. This process is continued infinitely. Prove that the sphereflake has an infinite surface area.

![Sphereflake Image]

109. Salary You go to work at a company that pays $0.01 for the first day, $0.02 for the second day, $0.04 for the third day, and so on. If the daily wage keeps doubling, what would your total income be for working (a) 29 days, (b) 30 days, and (c) 31 days?

110. Annuities When an employee receives a paycheck at the end of each month, $P$ dollars is invested in a retirement account. These deposits are made each month for $t$ years and the account earns interest at the annual percentage rate $r$. If the interest is compounded monthly, the amount $A$ in the account at the end of $t$ years is

$$A = P + P\left(1 + \frac{r}{12}\right)^{12t} + \ldots + P\left(1 + \frac{r}{12}\right)^{12(t-1)} = P\left(\frac{12}{r}\right)\left(1 + \frac{r}{12}\right)^{12t} - 1.$$ 

If the interest is compounded continuously, the amount $A$ in the account after $t$ years is

$$A = Pe^{rt} + Pe^{rt/12} + Pe^{rt/2} + Pe^{rt/3} + \ldots = Pe^{rt}(1 - \frac{r}{2}) = \frac{P(e^{rt} - 1)}{e^{rt} - 1}.$$ 

Verify the formulas for the sums given above.

Annuities In Exercises 111–114, consider making monthly deposits of $P$ dollars in a savings account at an annual interest rate $r$. Use the results of Exercise 110 to find the balance $A$ after $t$ years if the interest is compounded (a) monthly and (b) continuously.

111. $P = 500$, $r = 3\%$, $t = 20$ years
112. $P = 750$, $r = 5\%$, $t = 25$ years
113. $P = 1000$, $r = 4\%$, $t = 40$ years
114. $P = 200$, $r = 6\%$, $t = 50$ years

115. Modeling Data The annual sales $a_n$ (in millions of dollars) for Avon Products, Inc. from 1993 through 2002 are given below as ordered pairs of the form $(n, a_n)$, where $n$ represents the year, with $n = 3$ corresponding to 1993. (Source: 2002 Avon Products, Inc. Annual Report)

\[(3, 3844), (4, 4267), (5, 4492), (6, 4814), (7, 5079), (8, 5213), (9, 5289), (10, 5682), (11, 5958), (12, 6171)\]

(a) Use the regression capabilities of a graphing utility to find a model of the form

$$a_n = ce^{rn}, \quad n = 3, 4, 5, \ldots, 12$$

for the data. Graphically compare the points and the model.

(b) Use the data to find the total sales for the 10-year period.

(c) Approximate the total sales for the 10-year period using the formula for the sum of a geometric series. Compare the result with that in part (b).

116. Salary You accept a job that pays a salary of $40,000 for the first year. During the next 39 years you receive a 4% raise each year. What would be your total compensation over the 40-year period?

True or False? In Exercises 117–122, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

117. If $\lim_{n \to \infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

118. If $\sum_{n=1}^{\infty} a_n = L$, then $\sum_{n=0}^{\infty} a_n = L + a_0$.

119. If $|r| < 1$, then $\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$.

120. The series $\sum_{n=1}^{\infty} \frac{n}{1000(n+1)}$ diverges.

121. 0.75 = 0.749999 . . .

122. Every decimal with a repeating pattern of digits is a rational number.

123. Show that the series $\sum_{n=1}^{\infty} c_n$ can be written in the telescoping form

$$\sum_{n=1}^{\infty} [(c - S_{n-1}) - (c - S_n)]$$

where $S_0 = 0$ and $S_n$ is the nth partial sum.

124. Let $\sum a_n$ be a convergent series, and let $R_N = a_{N+1} + a_{N+2} + \cdots$ be the remainder of the series after the first $N$ terms. Prove that $\lim_{N \to \infty} R_N = 0$.

125. Find two divergent series $\sum a_n$ and $\sum b_n$ such that $\sum(a_n + b_n)$ converges.

126. Given two infinite series $\sum a_n$ and $\sum b_n$ such that $\sum a_n$ converges and $\sum b_n$ diverges, prove that $\sum(a_n + b_n)$ diverges.

127. Suppose that $\sum a_n$ diverges and $c$ is a nonzero constant. Prove that $\sum ca_n$ diverges.
128. If \( \sum_{n=1}^{\infty} a_n \) converges where \( a_n \) is nonzero, show that \( \sum_{n=1}^{\infty} \frac{1}{a_n} \) diverges.

129. The Fibonacci sequence is defined recursively by \( a_{n+2} = a_n + a_{n+1} \), where \( a_1 = 1 \) and \( a_2 = 1 \).
(a) Show that \( \frac{1}{a_{n+1} a_{n+2}} = \frac{1}{a_{n+1}} - \frac{1}{a_{n+2}} a_{n+3} \).
(b) Show that \( \sum_{n=0}^{\infty} \frac{1}{a_{n+1} a_{n+3}} = 1 \).

130. Find the values of \( x \) for which the infinite series \( 1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + x^6 + \cdots \) converges. What is the sum when the series converges?

131. Prove that \( \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots = \frac{1}{r-1} \), for \( |r| > 1 \).

132. Writing The figure below represents an informal way of showing that \( \sum_{n=1}^{\infty} \frac{1}{n^2} < 2 \). Explain how the figure implies this conclusion.

![Diagram of a figure representing an informal way of showing the sum of the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).]

**FOR FURTHER INFORMATION** For more on this exercise, see the article “Convergence with Pictures” by P.J. Rippon in *American Mathematical Monthly.*

133. Writing Read the article “The Exponential-Decay Law Applied to Medical Dosages” by Gerald M. Armstrong and Calvin P. Midgley in *Mathematics Teacher.* (To view this article, go to the website [www.matharticles.com](http://www.matharticles.com).) Then write a paragraph on how a geometric sequence can be used to find the total amount of a drug that remains in a patient’s system after \( n \) equal doses have been administered (at equal time intervals).

**Putnam Exam Challenge**

134. Write \( \sum_{n=1}^{\infty} \frac{4^n}{(3^n+1)(3^n-2)} \) as a rational number.

135. Let \( f(n) \) be the sum of the first \( n \) terms of the sequence 0, 1, 1, 2, 2, 3, 3, 4, . . . , where the \( n \)th term is given by
\[
a_n = \begin{cases} 
 n/2, & \text{if } n \text{ is even} \\
 (n - 1)/2, & \text{if } n \text{ is odd}.
\end{cases}
\]
Show that if \( x \) and \( y \) are positive integers and \( x > y \) then \( xy = f(x + y) - f(x - y) \).

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