Section 9.4 Comparisons of Series

- Use the Direct Comparison Test to determine whether a series converges or diverges.
- Use the Limit Comparison Test to determine whether a series converges or diverges.

**Direct Comparison Test**

For the convergence tests developed so far, the terms of the series have to be fairly simple and the series must have special characteristics in order for the convergence tests to be applied. A slight deviation from these special characteristics can make a test nonapplicable. For example, in the following pairs, the second series cannot be tested by the same convergence test as the first series even though it is similar to the first.

1. \( \sum_{n=0}^{\infty} \frac{1}{7^n} \) is geometric, but \( \sum_{n=0}^{\infty} \frac{n}{7^n} \) is not.
2. \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is a p-series, but \( \sum_{n=1}^{\infty} \frac{1}{n^3 + 1} \) is not.
3. \( a_n = \frac{n}{(n^2 + 3)^2} \) is easily integrated, but \( b_n = \frac{n^2}{(n^2 + 3)^2} \) is not.

In this section you will study two additional tests for positive-term series. These two tests greatly expand the variety of series you are able to test for convergence or divergence. They allow you to **compare** a series having complicated terms with a simpler series whose convergence or divergence is known.

**THEOREM 9.12 Direct Comparison Test**

Let \( 0 < a_n \leq b_n \) for all \( n \).

1. If \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges.
2. If \( \sum_{n=1}^{\infty} a_n \) diverges, then \( \sum_{n=1}^{\infty} b_n \) diverges.

**Proof** To prove the first property, let \( L = \sum_{n=1}^{\infty} b_n \) and let

\[
S_n = a_1 + a_2 + \cdots + a_n.
\]

Because \( 0 < a_n \leq b_n \), the sequence \( S_1, S_2, S_3, \ldots \) is nondecreasing and bounded above by \( L \); so, it must converge. Because

\[
\lim_{n \to \infty} S_n = \sum_{n=1}^{\infty} a_n,
\]

it follows that \( \sum a_n \) converges. The second property is logically equivalent to the first.

**NOTE** As stated, the Direct Comparison Test requires that \( 0 < a_n \leq b_n \) for all \( n \). Because the convergence of a series is not dependent on its first several terms, you could modify the test to require only that \( 0 < a_n \leq b_n \) for all \( n \) greater than some integer \( N \).
**EXAMPLE 1**  
Using the Direct Comparison Test

Determine the convergence or divergence of

\[ \sum_{n=1}^{\infty} \frac{1}{2 + 3^n} \]

**Solution**  
This series resembles

\[ \sum_{n=1}^{\infty} \frac{1}{3^n} \]  
Convergent geometric series

Term-by-term comparison yields

\[ a_n = \frac{1}{2 + 3^n} < \frac{1}{3^n} = b_n, \quad n \geq 1. \]

So, by the Direct Comparison Test, the series converges.

**EXAMPLE 2**  
Using the Direct Comparison Test

Determine the convergence or divergence of

\[ \sum_{n=1}^{\infty} \frac{1}{2 + \sqrt{n}} \]

**Solution**  
This series resembles

\[ \sum_{n=1}^{\infty} \frac{1}{n^{1/3}} \]  
Divergent $p$-series

Term-by-term comparison yields

\[ \frac{1}{2 + \sqrt{n}} \leq \frac{1}{\sqrt{n}}, \quad n \geq 1 \]

which does not meet the requirements for divergence. (Remember that if term-by-term comparison reveals a series that is smaller than a divergent series, the Direct Comparison Test tells you nothing.) Still expecting the series to diverge, you can compare the given series with

\[ \sum_{n=1}^{\infty} \frac{1}{n} \]  
Divergent harmonic series

In this case, term-by-term comparison yields

\[ a_n = \frac{1}{n} \leq \frac{1}{2 + \sqrt{n}} = b_n, \quad n \geq 4 \]

and, by the Direct Comparison Test, the given series diverges.

---

**NOTE**  
To verify the last inequality in Example 2, try showing that

\[ 2 + \sqrt{n} \leq n \quad \text{whenever} \quad n \geq 4. \]

Remember that both parts of the Direct Comparison Test require that $0 < a_n \leq b_n$. Informally, the test says the following about the two series with nonnegative terms.

1. If the “larger” series converges, the “smaller” series must also converge.
2. If the “smaller” series diverges, the “larger” series must also diverge.
**Limit Comparison Test**

Often a given series closely resembles a $p$-series or a geometric series, yet you cannot establish the term-by-term comparison necessary to apply the Direct Comparison Test. Under these circumstances you may be able to apply a second comparison test, called the **Limit Comparison Test**.

**THEOREM 9.13 Limit Comparison Test**

Suppose that $a_n > 0$, $b_n > 0$, and

$$\lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = L$$

where $L$ is finite and positive. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

**Proof** Because $a_n > 0$, $b_n > 0$, and

$$\lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = L$$

there exists $N > 0$ such that

$$0 < \frac{a_n}{b_n} < L + 1, \quad \text{for } n \geq N.$$  

This implies that

$$0 < a_n < (L + 1)b_n.$$  

So, by the Direct Comparison Test, the convergence of $\sum b_n$ implies the convergence of $\sum a_n$. Similarly, the fact that

$$\lim_{n \to \infty} \left( \frac{b_n}{a_n} \right) = \frac{1}{L}$$

can be used to show that the convergence of $\sum a_n$ implies the convergence of $\sum b_n$.

**EXAMPLE 3 Using the Limit Comparison Test**

Show that the following general harmonic series diverges.

$$\sum_{n=1}^{\infty} \frac{1}{an + b}, \quad a > 0, \quad b > 0$$

**Solution** By comparison with

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{Divergent harmonic series}$$

you have

$$\lim_{n \to \infty} \frac{1/(an + b)}{1/n} = \lim_{n \to \infty} \frac{n}{an + b} = \frac{1}{a}.$$  

Because this limit is greater than 0, you can conclude from the Limit Comparison Test that the given series diverges.
The Limit Comparison Test works well for comparing a “messy” algebraic series with a $p$-series. In choosing an appropriate $p$-series, you must choose one with an $n$th term of the same magnitude as the $n$th term of the given series.

<table>
<thead>
<tr>
<th>Given Series</th>
<th>Comparison Series</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$</td>
<td>$\sum_{n=1}^{\infty} \frac{1}{n^2}$</td>
<td>Both series converge.</td>
</tr>
<tr>
<td>$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n - 2}}$</td>
<td>$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$</td>
<td>Both series diverge.</td>
</tr>
<tr>
<td>$\sum_{n=1}^{\infty} \frac{n^2 - 10}{4n^3 + n^5}$</td>
<td>$\sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$</td>
<td>Both series converge.</td>
</tr>
</tbody>
</table>

In other words, when choosing a series for comparison, you can disregard all but the highest powers of $n$ in both the numerator and the denominator.

**EXAMPLE 4** Using the Limit Comparison Test

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$$

**Solution** Disregarding all but the highest powers of $n$ in the numerator and the denominator, you can compare the series with

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

Convergent $p$-series

Because

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( \frac{\sqrt{n}}{n^2 + 1} \right) \left( \frac{n^{3/2}}{1} \right)$$

$$= \lim_{n \to \infty} \frac{n^{3/2}}{n^2 + 1} = 1$$

you can conclude by the Limit Comparison Test that the given series converges.

**EXAMPLE 5** Using the Limit Comparison Test

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{n^{2n}}{4n^3 + 1}$$

**Solution** A reasonable comparison would be with the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n^3}$$

Divergent series

Note that this series diverges by the $n$th-Term Test. From the limit

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( \frac{n^{2n}}{4n^3 + 1} \right) \left( \frac{n^3}{2^n} \right)$$

$$= \lim_{n \to \infty} \frac{1}{4 + (1/n^3)} = \frac{1}{4}$$

you can conclude that the given series diverges.
Exercises for Section 9.4

1. Graphical Analysis  The figures show the graphs of the first 10 terms, and the graphs of the first 10 terms of the sequence of partial sums, of each series.

\[ \sum_{n=1}^{\infty} \frac{6}{n^{1/2}}, \quad \sum_{n=1}^{\infty} \frac{6}{n^{1/2} + 3}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{6}{n\sqrt{n} + 0.5} \]

(a) Identify the series in each figure.
(b) Which series is a p-series? Does it converge or diverge?
(c) For the series that are not p-series, how do the magnitudes of the terms compare with the magnitudes of the terms of the p-series? What conclusion can you draw about the convergence or divergence of the series?
(d) Explain the relationship between the magnitudes of the terms of the series and the magnitudes of the terms of the partial sums.

\[ a_n \quad \text{Graphs of terms} \]
\[ S_n \quad \text{Graphs of partial sums} \]

2. Graphical Analysis  The figures show the graphs of the first 10 terms, and the graphs of the first 10 terms of the sequence of partial sums, of each series.

\[ \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}, \quad \sum_{n=1}^{\infty} \frac{2}{\sqrt{n} - 0.5}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{4}{\sqrt{n} + 0.5} \]

(a) Identify the series in each figure.
(b) Which series is a p-series? Does it converge or diverge?
(c) For the series that are not p-series, how do the magnitudes of the terms compare with the magnitudes of the terms of the p-series? What conclusion can you draw about the convergence or divergence of the series?
(d) Explain the relationship between the magnitudes of the terms of the series and the magnitudes of the terms of the partial sums.

\[ a_n \quad \text{Graphs of terms} \]
\[ S_n \quad \text{Graphs of partial sums} \]

In Exercises 3–14, use the Direct Comparison Test to determine the convergence or divergence of the series.

3. \[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \]
4. \[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 2} \]
5. \[ \sum_{n=1}^{\infty} \frac{1}{n + 1} \]
6. \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n + 1}} \]
7. \[ \sum_{n=1}^{\infty} \frac{1}{3^n + 1} \]
8. \[ \sum_{n=1}^{\infty} \frac{1}{n + 1} \]
9. \[ \sum_{n=1}^{\infty} \ln n \]
10. \[ \sum_{n=1}^{\infty} \frac{1}{n!} \]
11. \[ \sum_{n=1}^{\infty} \frac{1}{4^n} \]
12. \[ \sum_{n=1}^{\infty} \frac{1}{3^n - 1} \]
13. \[ \sum_{n=1}^{\infty} e^{-n^2} \]
14. \[ \sum_{n=1}^{\infty} \frac{2}{3^n - 5} \]

In Exercises 15–28, use the Limit Comparison Test to determine the convergence or divergence of the series.

15. \[ \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \]
16. \[ \sum_{n=1}^{\infty} \frac{2}{3^n - 5} \]
17. \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + 1} \]
18. \[ \sum_{n=1}^{\infty} \frac{3}{\sqrt{n} - 4} \]
19. \[ \sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^3 + 2n + 1} \]
20. \[ \sum_{n=1}^{\infty} \frac{5n - 3}{n^2 - 2n + 5} \]
21. \[ \sum_{n=1}^{\infty} \frac{n + 3}{n(n + 2)} \]
22. \[ \sum_{n=1}^{\infty} \frac{1}{n(n + 1)} \]
23. \[ \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2 + 1}} \]
24. \[ \sum_{n=1}^{\infty} \frac{n}{(n + 1)2^{n-1}} \]
25. \[ \sum_{n=1}^{\infty} \frac{n^k}{n^k + 1} \]
26. \[ \sum_{n=1}^{\infty} \frac{5}{n + \sqrt{n^2 + 4}} \]
27. \[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \]
28. \[ \sum_{n=1}^{\infty} \frac{1}{n + 1} \]

In Exercises 29–36, test for convergence or divergence, using each test at least once. Identify which test was used.

(a) \text{nth-Term Test}  
(b) \text{Geometric Series Test}  
(c) \text{p-Series Test}  
(d) \text{Telescoping Series Test}  
(e) \text{Integral Test}  
(f) \text{Direct Comparison Test}  
(g) \text{Limit Comparison Test}

29. \[ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} \]
30. \[ \sum_{n=0}^{\infty} \frac{1}{3^n} \]
31. \[ \sum_{n=1}^{\infty} \frac{1}{3^n + 2} \]
32. \[ \sum_{n=1}^{\infty} \frac{1}{3n^2 - 2n - 15} \]
33. \[ \sum_{n=1}^{\infty} \frac{n}{2n + 3} \]
34. \[ \sum_{n=1}^{\infty} \frac{1}{n + 1} - \frac{1}{n + 2} \]
35. \[ \sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2} \]
36. \[ \sum_{n=1}^{\infty} \frac{3}{n(n^2 + 3)} \]
37. Use the Limit Comparison Test with the harmonic series to show that the series \( \sum a_n \) (where \( 0 < a_n < a_{n-1} \)) diverges if \( \lim_{n \to \infty} \frac{a_n}{n} \) is finite and nonzero.

38. Prove that, if \( P(n) \) and \( Q(n) \) are polynomials of degree \( j \) and \( k \), respectively, then the series

\[
\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}
\]

converges if \( j < k - 1 \) and diverges if \( j \geq k - 1 \).

In Exercises 39–42, use the polynomial test given in Exercise 38 to determine whether the series converges or diverges.

39. \( \frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{7} + \frac{1}{20} + \cdots \)

40. \( \frac{1}{2} + \frac{1}{13} + \frac{1}{3} + \frac{1}{25} + \cdots \)

41. \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \)

42. \( \sum_{n=1}^{\infty} \frac{n^3}{n^4 + 3} \)

In Exercises 43 and 44, use the divergence test given in Exercise 37 to show that the series diverges.

43. \( \sum_{n=1}^{\infty} \frac{n^3}{5n^4 + 3} \)

44. \( \sum_{n=1}^{\infty} \frac{1}{n\ln n} \)

In Exercises 45–48, determine the convergence or divergence of the series.

45. \( \frac{1}{30} + \frac{1}{36} + \frac{1}{120} + \frac{1}{132} + \cdots \)

46. \( \frac{1}{31} + \frac{1}{37} + \frac{1}{120} + \frac{1}{132} + \cdots \)

47. \( \frac{1}{39} + \frac{1}{37} + \frac{1}{120} + \frac{1}{132} + \cdots \)

48. \( \frac{1}{33} + \frac{1}{31} + \frac{1}{120} + \frac{1}{132} + \cdots \)

### Writing About Concepts

49. Review the results of Exercises 45–48. Explain why careful analysis is required to determine the convergence or divergence of a series and why only considering the magnitudes of the terms of a series could be misleading.

50. State the Direct Comparison Test and give an example of its use.

51. State the Limit Comparison Test and give an example of its use.

52. It appears that the terms of the series

\[
\frac{1}{1000} + \frac{1}{2000} + \frac{1}{3000} + \cdots
\]

are less than the corresponding terms of the convergent series

\[
1 + \frac{1}{2} + \frac{1}{3} + \cdots
\]

If the statement above is correct, the first series converges. Is this correct? Why or why not? Make a statement about how the divergence or convergence of a series is affected by inclusion or exclusion of the first finite number of terms.

53. The figure shows the first 20 terms of the convergent series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) and the first 20 terms of the series \( \sum_{n=1}^{\infty} \frac{1}{n} \). Identify the two series and explain your reasoning in making the selection.

54. Consider the series \( \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \).

(a) Verify that the series converges.

(b) Use a graphing utility to complete the table.

<table>
<thead>
<tr>
<th>n</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_n )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) The sum of the series is \( \pi^2/8 \). Find the sum of the series \( \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \).

(d) Use a graphing utility to find the sum of the series \( \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \).

**True or False?** In Exercises 55–60, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

55. If \( 0 < a_n \leq b_n \) and \( \sum_{n=1}^{\infty} a_n \) converges, then \( \sum_{n=1}^{\infty} b_n \) diverges.

56. If \( 0 < a_{n+10} \leq b_n \) and \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges.

57. If \( a_n + b_n \leq c_n \) and \( \sum_{n=1}^{\infty} c_n \) converges, then the series \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) both converge. (Assume that the terms of all three series are positive.)

58. If \( a_n \leq b_n + c_n \) and \( \sum_{n=1}^{\infty} a_n \) diverges, then the series \( \sum_{n=1}^{\infty} b_n \) and \( \sum_{n=1}^{\infty} c_n \) both diverge. (Assume that the terms of all three series are positive.)

59. If \( 0 < a_n \leq b_n \) and \( \sum_{n=1}^{\infty} a_n \) diverges, then \( \sum_{n=1}^{\infty} b_n \) diverges.
60. If \( 0 < a_n \leq b_n \) and \( \sum_{n=1}^{\infty} b_n \) diverges, then \( \sum_{n=1}^{\infty} a_n \) diverges.

61. Prove that if the nonnegative series
\[
\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n
\]
converge, then so does the series
\[
\sum_{n=1}^{\infty} a_n b_n.
\]

62. Use the result of Exercise 61 to prove that if the nonnegative series
\[
\sum_{n=1}^{\infty} a_n
\]
converges, then so does the series
\[
\sum_{n=1}^{\infty} a_n^2.
\]

63. Find two series that demonstrate the result of Exercise 61.

64. Find two series that demonstrate the result of Exercise 62.

65. Suppose that \( \sum a_n \) and \( \sum b_n \) are series with positive terms. Prove that if \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \) and \( \sum b_n \) converges, \( \sum a_n \) also converges.

66. Suppose that \( \sum a_n \) and \( \sum b_n \) are series with positive terms. Prove that if \( \lim_{n \to \infty} \frac{a_n}{b_n} = \infty \) and \( \sum b_n \) diverges, \( \sum a_n \) also diverges.

67. Use the result of Exercise 65 to show that each series converges.
(a) \( \sum_{n=1}^{\infty} \frac{1}{(n+1)^3} \)
(b) \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n} n^n} \)

68. Use the result of Exercise 66 to show that each series diverges.
(a) \( \sum_{n=1}^{\infty} \frac{\ln n}{n} \)
(b) \( \sum_{n=2}^{\infty} \frac{1}{\ln n} \)

69. Suppose that \( \sum a_n \) is a series with positive terms. Prove that if \( \sum a_n \) converges, then \( \sum \sin a_n \) also converges.

70. Prove that the series \( \sum_{n=1}^{\infty} \frac{1}{1 + 2 + 3 + \ldots + n} \) converges.

### Putnam Exam Challenge

71. Is the infinite series
\[
\sum_{n=1}^{\infty} \frac{1}{n^{n + 1/n}}
\]
convergent? Prove your statement.

72. Prove that if \( \sum \alpha_n \) is a convergent series of positive real numbers, then so is \( \sum (\alpha_n)^{n/(n+1)} \).

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**Section Project: Solera Method**

Most wines are produced entirely from grapes grown in a single year. Sherry, however, is a complex mixture of older wines with new wines. This is done with a sequence of barrels (called a solera) stacked on top of each other, as shown in the photo.

Everton/The Image Works

The oldest wine is in the bottom tier of barrels, and the newest is in the top tier. Each year, half of each barrel in the bottom tier is bottled as sherry. The bottom barrels are then refilled with the wine from the barrels above. This process is repeated throughout the solera, with new wine being added to the top barrels. A mathematical model for the amount of \( n \)-year-old wine that is removed from a solera (with \( k \) tiers) each year is

\[
f(n, k) = \left( \frac{n - 1}{k - 1} \right) \left( \frac{1}{2} \right)^{n+1}, \quad k \leq n.
\]

(a) Consider a solera that has five tiers, numbered \( k = 1, 2, 3, 4, \) and 5. In 1990 \( (n = 0) \), half of each barrel in the top tier (tier 1) was refilled with new wine. How much of this wine was removed from the solera in 1991? In 1992? In 1993? . . . In 2005? During which year(s) was the greatest amount of the 1990 wine removed from the solera?

(b) In part (a), let \( a_n \) be the amount of 1990 wine that is removed from the solera in year \( n \). Evaluate
\[
\sum_{n=0}^{\infty} a_n.
\]

**FOR FURTHER INFORMATION** See the article “Finding Vintage Concentrations in a Sherry Solera” by Rhodes Peele and John T. MacQueen in the *UMAP Modules*. 