The Ratio and Root Tests

- Use the Ratio Test to determine whether a series converges or diverges.
- Use the Root Test to determine whether a series converges or diverges.
- Review the tests for convergence and divergence of an infinite series.

The Ratio Test

This section begins with a test for absolute convergence—the Ratio Test.

**THEOREM 9.17 Ratio Test**

Let $\sum a_n$ be a series with nonzero terms.

1. $\sum a_n$ converges absolutely if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$.
2. $\sum a_n$ diverges if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ or $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$.
3. The Ratio Test is inconclusive if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

**Proof** To prove Property 1, assume that

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = r < 1$$

and choose $R$ such that $0 \leq r < R < 1$. By the definition of the limit of a sequence, there exists some $N > 0$ such that $\left| \frac{a_{n+1}}{a_n} \right| < R$ for all $n > N$. Therefore, you can write the following inequalities.

$$|a_{N+1}| < |a_N|R$$
$$|a_{N+2}| < |a_{N+1}|R < |a_N|R^2$$
$$|a_{N+3}| < |a_{N+2}|R < |a_{N+1}|R^2 < |a_N|R^3$$

The geometric series $\sum |a_N|R^n = |a_N|R + |a_N|R^2 + \ldots + |a_N|R^n + \ldots$ converges, and so, by the Direct Comparison Test, the series

$$\sum_{n=1}^{\infty} |a_{n+1}| = |a_{N+1}| + |a_{N+2}| + \ldots + |a_{N+n}| + \ldots$$

also converges. This in turn implies that the series $\sum |a_n|$ converges, because discarding a finite number of terms ($n = N - 1$) does not affect convergence. Consequently, by Theorem 9.16, the series $\sum a_n$ converges absolutely. The proof of Property 2 is similar and is left as an exercise (see Exercise 98).

NOTE The fact that the Ratio Test is inconclusive when $\left| \frac{a_{n+1}}{a_n} \right| \to 1$ can be seen by comparing the two series $\sum (1/n)$ and $\sum (1/n^2)$. The first series diverges and the second one converges, but in both cases

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1.$$
Although the Ratio Test is not a cure for all ills related to tests for convergence, it is particularly useful for series that converge rapidly. Series involving factorials or exponentials are frequently of this type.

**EXAMPLE 1 Using the Ratio Test**

Determine the convergence or divergence of

\[ \sum_{n=0}^{\infty} \frac{2^n}{n!} \]

**Solution** Because \( a_n = \frac{2^n}{n!} \), you can write the following.

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n \to \infty} \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{2}{n+1} = 0
\]

Therefore, the series converges.

**EXAMPLE 2 Using the Ratio Test**

Determine whether each series converges or diverges.

a. \( \sum_{n=0}^{\infty} \frac{n^{2n+1}}{3^n} \)  
b. \( \sum_{n=1}^{\infty} \frac{n^n}{n!} \)

**Solution**

a. This series converges because the limit of \( |a_{n+1}/a_n| \) is less than 1.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^{2(n+1)}}{3^{n+1}} \cdot \frac{3^n}{n^{2n+1}} \right| = \lim_{n \to \infty} \frac{2^{n+1}}{3n^2} = \frac{2}{3} < 1
\]

b. This series diverges because the limit of \( |a_{n+1}/a_n| \) is greater than 1.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right| = \lim_{n \to \infty} \frac{(n+1)^n}{n^n} = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e > 1
\]
EXAMPLE 3  **A Failure of the Ratio Test**

Determine the convergence or divergence of \( \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1} \).

**Solution**  The limit of \( \frac{a_{n+1}}{a_n} \) is equal to 1.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left[ \frac{\sqrt{n+1}}{n+2} \left( \frac{n+1}{\sqrt{n}} \right) \right] = \lim_{n \to \infty} \left[ \sqrt{\frac{n+1}{n}} \left( \frac{n+1}{n+2} \right) \right] = \sqrt{1} = 1
\]

So, the Ratio Test is inconclusive. To determine whether the series converges, you need to try a different test. In this case, you can apply the Alternating Series Test. To show that let

\[ f(x) = \frac{\sqrt{x}}{x+1} \]

Then the derivative is

\[ f'(x) = \frac{-x + 1}{2\sqrt{x(x+1)^2}} \]

Because the derivative is negative for \( x > 1 \), you know that \( f \) is a decreasing function. Also, by L'Hôpital's Rule,

\[
\lim_{x \to \infty} \frac{\sqrt{x}}{x+1} = \lim_{x \to \infty} \frac{1/(2\sqrt{x})}{1} = \lim_{x \to \infty} \frac{1}{2\sqrt{x}} = 0.
\]

Therefore, by the Alternating Series Test, the series converges.

The series in Example 3 is conditionally convergent. This follows from the fact that the series

\[ \sum_{n=1}^{\infty} |a_n| \]

diverges (by the Limit Comparison Test with \( \sum 1/\sqrt{n} \)), but the series

\[ \sum_{n=1}^{\infty} a_n \]

converges.

**TECHNOLOGY**  A computer or programmable calculator can reinforce the conclusion that the series in Example 3 converges conditionally. By adding the first 100 terms of the series, you obtain a sum of about \(-0.2\). (The sum of the first 100 terms of the series \( \sum |a_n| \) is about 17.)
The Root Test

The next test for convergence or divergence of series works especially well for series involving \( n \)th powers. The proof of this theorem is similar to that given for the Ratio Test, and is left as an exercise (see Exercise 99).

**EXAMPLE 4  Using the Root Test**

Determine the convergence or divergence of

\[
\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}.
\]

**Solution**

You can apply the Root Test as follows.

\[
\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \frac{e^{2n}/n^n}{n^{n/n}} = \lim_{n \to \infty} \frac{e^2}{n} = 0 < 1
\]

Because this limit is less than 1, you can conclude that the series converges absolutely (and therefore converges).

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**FOR FURTHER INFORMATION** For more information on the usefulness of the Root Test, see the article “\( N! \) and the Root Test” by Charles C. Mumma II in *The American Mathematical Monthly*. To view this article, go to the website www.matharticles.com.
Strategies for Testing Series

You have now studied 10 tests for determining the convergence or divergence of an infinite series. (See the summary in the table on page 644.) Skill in choosing and applying the various tests will come only with practice. Below is a set of guidelines for choosing an appropriate test.

Guidelines for Testing a Series for Convergence or Divergence

1. Does the $n$th term approach 0? If not, the series diverges.
2. Is the series one of the special types—geometric, $p$-series, telescoping, or alternating?
3. Can the Integral Test, the Root Test, or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

In some instances, more than one test is applicable. However, your objective should be to learn to choose the most efficient test.

**EXAMPLE 5** Applying the Strategies for Testing Series

Determine the convergence or divergence of each series.

a. $\sum_{n=1}^{\infty} \frac{n + 1}{3n + 1}$  
   b. $\sum_{n=1}^{\infty} \left( \frac{\pi}{6} \right)^n$  
   c. $\sum_{n=1}^{\infty} ne^{-n^2}$

d. $\sum_{n=1}^{\infty} \frac{1}{3n + 1}$  
   e. $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n + 1}$  
   f. $\sum_{n=1}^{\infty} \frac{n^4}{10^n}$

g. $\sum_{n=1}^{\infty} \left( \frac{n + 1}{2n + 1} \right)^n$

**Solution**

a. For this series, the limit of the $n$th term is not 0 ($a_n \to \frac{1}{3}$ as $n \to \infty$). So, by the $n$th-Term Test, the series diverges.

b. This series is geometric. Moreover, because the ratio $r = \frac{\pi}{6}$ of the terms is less than 1 in absolute value, you can conclude that the series converges.

c. Because the function $f(x) = xe^{-x^2}$ is easily integrated, you can use the Integral Test to conclude that the series converges.

d. The $n$th term of this series can be compared to the $n$th term of the harmonic series. After using the Limit Comparison Test, you can conclude that the series diverges.

e. This is an alternating series whose $n$th term approaches 0. Because $a_{n+1} \leq a_n$, you can use the Alternating Series Test to conclude that the series converges.

f. The $n$th term of this series involves a factorial, which indicates that the Ratio Test may work well. After applying the Ratio Test, you can conclude that the series diverges.

g. The $n$th term of this series involves a variable that is raised to the $n$th power, which indicates that the Root Test may work well. After applying the Root Test, you can conclude that the series converges.
# Summary of Tests for Series

<table>
<thead>
<tr>
<th>Test</th>
<th>Series</th>
<th>Condition(s) of Convergence</th>
<th>Condition(s) of Divergence</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>nth-Term</strong></td>
<td>$\sum_{n=1}^{\infty} a_n$</td>
<td>$\lim_{n \to \infty} a_n \neq 0$</td>
<td></td>
<td>This test cannot be used to show convergence.</td>
</tr>
<tr>
<td><strong>Geometric Series</strong></td>
<td>$\sum_{n=0}^{\infty} ar^n$</td>
<td>$</td>
<td>r</td>
<td>&lt; 1$</td>
</tr>
<tr>
<td><strong>Telescoping Series</strong></td>
<td>$\sum_{n=1}^{\infty} (b_n - b_{n+1})$</td>
<td>$\lim_{n \to \infty} b_n = L$</td>
<td></td>
<td>Sum: $S = b_1 - L$</td>
</tr>
<tr>
<td><strong>p-Series</strong></td>
<td>$\sum_{n=1}^{\infty} \frac{1}{n^p}$</td>
<td>$p &gt; 1$</td>
<td>$p \leq 1$</td>
<td></td>
</tr>
<tr>
<td><strong>Alternating Series</strong></td>
<td>$\sum_{n=1}^{\infty} (-1)^{n-1}a_n$</td>
<td>$0 &lt; a_{n+1} \leq a_n$ and $\lim_{n \to \infty} a_n = 0$</td>
<td></td>
<td>Remainder: $</td>
</tr>
<tr>
<td><strong>Integral</strong></td>
<td>$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$</td>
<td>$\int_1^{\infty} f(x) , dx$ converges</td>
<td>$\int_1^{\infty} f(x) , dx$ diverges</td>
<td>Remainder: $0 &lt; R_N &lt; \int_N^{\infty} f(x) , dx$</td>
</tr>
<tr>
<td><strong>Root</strong></td>
<td>$\sum_{n=1}^{\infty} a_n$</td>
<td>$\lim_{n \to \infty} \sqrt[n]{a_n} &lt; 1$</td>
<td>$\lim_{n \to \infty} \sqrt[n]{a_n} &gt; 1$</td>
<td>Test is inconclusive if $\lim_{n \to \infty} \sqrt[n]{a_n} = 1$.</td>
</tr>
<tr>
<td><strong>Ratio</strong></td>
<td>$\sum_{n=1}^{\infty} a_n$</td>
<td>$\lim_{n \to \infty} \left</td>
<td>\frac{a_{n+1}}{a_n} \right</td>
<td>&lt; 1$</td>
</tr>
<tr>
<td><strong>Direct Comparison</strong></td>
<td>$\sum_{n=1}^{\infty} a_n$, $(a_{n}, b_{n} &gt; 0)$</td>
<td>$0 &lt; a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges</td>
<td>$0 &lt; b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges</td>
<td></td>
</tr>
<tr>
<td><strong>Limit Comparison</strong></td>
<td>$\sum_{n=1}^{\infty} a_n$, $(a_{n}, b_{n} &gt; 0)$</td>
<td>$\lim_{n \to \infty} \frac{a_n}{b_n} = L &gt; 0$ and $\sum_{n=1}^{\infty} b_n$ converges</td>
<td>$\lim_{n \to \infty} \frac{a_n}{b_n} = L &gt; 0$ and $\sum_{n=1}^{\infty} b_n$ diverges</td>
<td></td>
</tr>
</tbody>
</table>
In Exercises 1–4, verify the formula.

1. \( \frac{(n+1)!}{(n-2)!} = (n+1)(n)(n-1) \)
2. \( \frac{(2k-2)!}{(2k)!} = \frac{1}{(2k)(2k-1)} \)
3. \( 1 \cdot 3 \cdot 5 \cdots (2k-1) = \frac{(2k)!}{2^k k!} \)
4. \( \frac{1}{1 \cdot 3 \cdot 5 \cdots (2k-5)} = \frac{2^k k!}{(2k-3)(2k-1)} \), \( k \geq 3 \)

In Exercises 5–10, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

(a) \( S_n \)
(b) \( S_n \)
(c) \( S_n \)
(d) \( S_n \)
(e) \( S_n \)
(f) \( S_n \)

**Numerical, Graphical, and Analytic Analysis**

In Exercises 11 and 12, (a) verify that the series converges. (b) Use a graphing utility to find the indicated partial sum \( S_n \) and complete the table. (c) Use a graphing utility to graph the first 10 terms of the sequence of partial sums. (d) Use the table to estimate the sum of the series. (e) Explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.

<table>
<thead>
<tr>
<th>( n )</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_n )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In Exercises 13–32, use the Ratio Test to determine the convergence or divergence of the series.

13. \( \sum_{n=0}^{\infty} \frac{n!}{3^n} \)
14. \( \sum_{n=0}^{\infty} \frac{3^n}{n!} \)
15. \( \sum_{n=1}^{\infty} n \left( \frac{3}{4} \right)^n \)
16. \( \sum_{n=1}^{\infty} n \left( \frac{1}{2} \right)^n \)
17. \( \sum_{n=1}^{\infty} \frac{n}{2^n} \)
18. \( \sum_{n=1}^{\infty} \frac{n^3}{2^n} \)
19. \( \sum_{n=1}^{\infty} \frac{2^n}{n^2} \)
20. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)} \)
21. \( \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n!} \)
22. \( \sum_{n=1}^{\infty} \frac{(-1)^n (3/2)^n}{n^2} \)
23. \( \sum_{n=1}^{\infty} \frac{n!}{n^3} \)
24. \( \sum_{n=1}^{\infty} \frac{(2n)!}{n^3} \)
25. \( \sum_{n=1}^{\infty} \frac{4^n}{n!} \)
26. \( \sum_{n=1}^{\infty} \frac{n^n}{n!} \)
27. \( \sum_{n=0}^{\infty} \frac{3^n}{(n+1)^3} \)
28. \( \sum_{n=0}^{\infty} \frac{3^n}{(n+1)!} \)
29. \( \sum_{n=0}^{\infty} \frac{4^n}{3^n + 1} \)
30. \( \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} \)
31. \( \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \)
32. \( \sum_{n=1}^{\infty} \frac{(-1)^n [2 \cdot 4 \cdot 6 \cdots (2n)]}{2 \cdot 5 \cdot 8 \cdots (3n-1)} \)

In Exercises 33–36, verify that the Ratio Test is inconclusive for the \( p \)-series.

33. \( \sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \)
34. \( \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \)
35. \( \sum_{n=1}^{\infty} \frac{1}{n^{1/3}} \)
36. \( \sum_{n=1}^{\infty} \frac{1}{n^{1/4}} \)
In Exercises 93–96, determine the convergence or divergence of the series.

93. \[ \sum_{n=2}^{\infty} \frac{n^3}{n^2 + 1} \]
94. \[ \sum_{n=2}^{\infty} \frac{2^n}{n} \]
95. \[ \sum_{n=2}^{\infty} \frac{n}{n^2 - 1} \]
96. \[ \sum_{n=2}^{\infty} \frac{n}{n^2 + 1} \]

In Exercises 97–100, determine the number of terms required to approximate the sum of the series with an error less than 0.0001.

97. \[ \sum_{n=1}^{k} \frac{1}{n} \]
98. \[ \sum_{n=1}^{k} \frac{1}{n^2} \]
99. \[ \sum_{n=1}^{k} \frac{1}{n^3} \]
100. \[ \sum_{n=1}^{k} \frac{1}{n^4} \]

In Exercises 101–104, determine the convergence or divergence of the series using any appropriate test from this chapter.

101. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \]
102. \[ \sum_{n=1}^{\infty} \frac{2^n}{n^3} \]
103. \[ \sum_{n=1}^{\infty} \frac{3}{n^2} \]
104. \[ \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \]

In Exercises 105–108, determine the convergence or divergence of the series recursively. Determine the number of terms required to approximate the sum of the series with an error less than 0.0001.

105. \[ a_1 = 1, a_{n+1} = \frac{1}{n} a_n \]
106. \[ a_1 = 2, a_{n+1} = \frac{1}{n} a_n \]
107. \[ a_1 = 3, a_{n+1} = \frac{1}{n} a_n \]
108. \[ a_1 = 4, a_{n+1} = \frac{1}{n} a_n \]

In Exercises 109–112, determine the convergence or divergence of the series.

109. \[ \sum_{n=1}^{\infty} \frac{n}{n+1} \]
110. \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \]
111. \[ \sum_{n=1}^{\infty} \frac{1}{n^3} \]
112. \[ \sum_{n=1}^{\infty} \frac{1}{n^4} \]
In Exercises 87–92, find the values of \( x \) for which the series converges.

87. \( \sum_{n=0}^{\infty} \left( \frac{x}{3} \right)^n \)
88. \( \sum_{n=0}^{\infty} \left( \frac{x+1}{4} \right)^n \)
89. \( \sum_{n=0}^{\infty} \frac{(-1)^n(x + 1)^n}{n} \)
90. \( \sum_{n=0}^{\infty} \frac{2(x - 1)^n}{n} \)
91. \( \sum_{n=0}^{\infty} \frac{n!(x/2)^n}{n!} \)
92. \( \sum_{n=0}^{\infty} \frac{(x + 1)^n}{n!} \)

### Writing About Concepts

93. State the Ratio Test.
94. State the Root Test.
95. You are told that the terms of a positive series appear to approach zero rapidly as \( n \) approaches infinity. In fact, \( a_n \leq 0.0001 \). Given no other information, does this imply that the series converges? Support your conclusion with examples.
96. The graph shows the first 10 terms of the sequence of partial sums of the convergent series
\( \sum_{n=0}^{\infty} \left( \frac{2n}{3n + 2} \right)^n \).
Find a series such that the terms of its sequence of partial sums are less than the corresponding terms of the sequence in the figure, but such that the series diverges. Explain your reasoning.

![Graph showing sequence](image)

97. Using the Ratio Test, it is determined that an alternating series converges. Does the series converge conditionally or absolutely? Explain.


99. Prove Theorem 9.18. (Hint for Property 1: If the limit equals \( r \), choose a real number \( R \) such that \( r < R < 1 \). By the definitions of the limit, there exists some \( N > 0 \) such that \( \lim_{n \to \infty} \frac{r}{|a_n|} < R \) for \( n > N \).)

100. Show that the Root Test is inconclusive for the \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \).
101. Show that the Ratio Test and the Root Test are both inconclusive for the logarithmic \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p} \).
102. Determine the convergence or divergence of the series \( \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \)
when (a) \( x = 1 \), (b) \( x = 2 \), (c) \( x = 3 \), and (d) \( x \) is a positive integer.
103. Show that if \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent, then
\( \left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n| \).
104. Writing Read the article “A Differentiation Test for Absolute Convergence” by Yaser S. Abu-Mostafa in *Mathematics Magazine*. Then write a paragraph that describes the test. Include examples of series that converge and examples of series that diverge.

### Putnam Exam Challenge

105. Is the following series convergent or divergent?
\[ 1 + \frac{1}{2} \cdot \frac{19}{7} + \frac{2!/(19)^2}{3!/(5^3)(7)} + \frac{3!/(19)^3}{4!/(4^4)(7)} + \frac{4!/(19)^4}{5!/(5^5)(7)} + \cdots \]
106. Show that if the series
\[ a_1 + a_2 + a_3 + \cdots + a_n + \cdots \]
converges, then the series
\[ a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} + \cdots \]
converges also.

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