Throughout this packet — there will be blanks you are expected to fill in prior to coming to class. This packet follows your Larson Textbook. Do NOT throw away! Keep in 3 ring-binder until the end of the course.

Chapter 5.1 The Natural Logarithmic Function: Differentiation

Mathematician: ________________

Definition of Natural Logarithm Function

The natural logarithmic function is defined by

$$\ln x = \int_1^x \frac{1}{t} \, dt, \quad x > 0.$$ 

The domain of the natural logarithmic function is the set of all positive real numbers.

Properties of Logarithms

The domain is \((0, \infty)\) and the range is \((-\infty, \infty)\).

The function is continuous, increasing, and one-to-one.

The graph is concave downward.

- \(\ln(1) = 0\)
- \(\ln(ab) = \ln a + \ln b\)
- \(\ln(a^n) = n \ln a\)
- \(\ln\left(\frac{a}{b}\right) = \ln a - \ln b\)

The Definition of e

The letter \(e\) denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t} \, dt = 1.$$
The Derivative of Natural Logarithmic Function

Let \( u \) be a differentiable function of \( x \).

1. \[
\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0
\]
2. \[
\frac{d}{dx}[\ln u] = \frac{1}{u} \frac{du}{dx} = \frac{u'}{u}, \quad u > 0
\]

If \( u \) is a differentiable function of \( x \) such that \( u \neq 0 \), then

\[
\frac{d}{dx}[\ln|u|] = \frac{u'}{u}.
\]

Why Logarithmic Differentiation???? Log Properties simplify derivative, have function in exponent

Steps Logarithmic Differentiation

\[
y = \frac{(x-2)^2}{\sqrt{x^2+1}}; \quad x \neq 2
\]

Step 1: Write original equation

\[
y = \frac{(x-2)^2}{\sqrt{x^2+1}}
\]

Step 2: Take natural log of each side

\[
\ln y = \ln \left( \frac{(x-2)^2}{\sqrt{x^2+1}} \right)
\]

Step 3: Use logarithmic properties to expand

\[
\ln y = 2 \ln(x-2) - \frac{1}{2} \left( x^2 + 1 \right)
\]

Step 4: Implicit Differentiate

\[
y' = 2 \left( \frac{2}{x-2} - \frac{1}{2} \cdot \frac{2x}{x^2+1} \right)
\]

Step 5: Solve for \( y' \)

\[
y' = y \left( \frac{2}{x-2} - \frac{x}{x^2+1} \right)
\]

Step 6: If possible, substitute for \( y \)

\[
y' = \left( \frac{(x-2)^2}{\sqrt{x^2+1}} \right) \left( \frac{2}{x-2} - \frac{x}{x^2+1} \right)
\]

Step 7: Simplify

\[
y' = \frac{(x-2)(x^2+2x+2)}{(x^2+1)^{3/2}}
\]
5.2 The Natural Logarithmic Function: Integration

Log Rule for Integration

1. \( \int \frac{1}{x} \, dx = \ln|x| + C \)
2. \( \int \frac{1}{u} \, du = \ln|u| + C \)

Alternate form of Log Rule:

Integrals of Trig Functions:

\[ \int \sin u \, du = -\cos u + C \]
\[ \int \cos u \, du = \sin u + C \]
\[ \int \tan u \, du = \int \frac{\sin u}{\cos u} \, du = -\ln|\cos u| + C \]
\[ \int \cot u \, du = \int \frac{\cos u}{\sin u} \, du = \ln|\sin u| + C \]
\[ \int \sec u \, du = \ln|\sec u + \tan u| + C \]
\[ \int \csc u \, du = -\ln|\csc u + \cot u| + C \]
5.3 Inverse Functions

Definition of Inverse:
A function \( g \) is the inverse function of the function \( f \) if
\[
 f(g(x)) = x \quad \text{for each } x \text{ in the domain of } g
\]
and
\[
 g(f(x)) = x \quad \text{for each } x \text{ in the domain of } f.
\]
The function \( g \) is denoted by \( f^{-1} \) (read “\( f \) inverse”).

Reflexive Properties

The graph of \( f \) contains the point \((a, b)\) if and only if the graph of \( f^{-1} \) contains the point \((b, a)\).

Existence of Inverse

1. A function has an inverse function if and only if it is one-to-one.
2. If \( f \) is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

Steps for finding inverse

Step 1: Verify an inverse exists
Step 2: Solve for \( x = g(y) = f^{-1}(x) \)
Step 3: Exchange \( x \) and \( y \) \((y = f^{-1}(x))\)
Step 4: Verify \( f(f^{-1}(x)) = x \) and \( f^{-1}(f(x)) = x \)

Continuity and Differentiability of Inverse:

Let \( f \) be a function whose domain is an interval \( I \). If \( f \) has an inverse function, then the following statements are true.

1. If \( f \) is continuous on its domain, then \( f^{-1} \) is continuous on its domain.
2. If \( f \) is increasing on its domain, then \( f^{-1} \) is increasing on its domain.
3. If \( f \) is decreasing on its domain, then \( f^{-1} \) is decreasing on its domain.
4. If \( f \) is differentiable at \( c \) and \( f'(c) \neq 0 \), then \( f^{-1} \) is differentiable at \( f(c) \).
Derivative of Inverse

Let \( f \) be a function that is differentiable on an interval \( I \). If \( f \) has an inverse function \( g \), then \( g \) is differentiable at any \( x \) for which \( f'(g(x)) \neq 0 \). Moreover,

\[
g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.
\]

\[
(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}
\]

Steps for finding inverse value at a point

Find \( f^{-1}(x) \) at \( x = 3 \) when \( f(x) = \frac{1}{4}x^3 + x - 1 \)

Step 1: Set \( f(x) = x - \) value

\[ f(3) = \frac{1}{4}x^3 + x - 1 \]

Step 2: Solve for \( x \)

\[ x = 2 \]

Step 3: Identify Function \((x, f(x))\)

Function: \((2, 3)\)

Step 4: Use Inverse Relationship to identify inverse

Inverse point: \((3, 2)\)

\( f^{-1}(3) = 2 \)

Steps for finding derivative of an inverse function at a point \( x = c \)

Find \( (f^{-1})'(x) \) at \( x = 3 \) when \( f(x) = \frac{1}{4}x^3 + x - 1 \)

Step 1: Calculate \( f^{-1}(x) \) at \( x \)

\[ f^{-1}(3) = 2 \] (Above)

Step 2: Calculate \( f'(x) \)

\[ f'(x) = \frac{3}{4}x^2 + 1 \]

Step 3: Evaluate \( f'(x) \) at \( f^{-1}(x) \) :: \( f'(f^{-1}(x)) \)

\[ f'(2) = \frac{3}{4}(2)^2 + 1 = 4 \]

Step 4: Evaluate \( f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \)

\[ (f^{-1})'(3) = \frac{1}{4} \]
5.4 Exponential Function: Differentiation and Integration

Definition of the Natural Exponential Function

The inverse function of the natural logarithmic function \( f(x) = \ln x \) is the **natural exponential function** and is denoted by

\[
f^{-1}(x) = e^x.
\]

That is,

\[
y = e^x \quad \text{if and only if} \quad x = \ln y.
\]

**Inverse:** \( \ln(e^x) = x \) and \( e^{\ln x} = x \)

Properties:

The domain of \( f(x) = e^x \) is \( (-\infty, \infty) \), and the range is \( (0, \infty) \).

The function \( f(x) = e^x \) is continuous, increasing, and one-to-one on its entire domain.

The graph of \( f(x) = e^x \) is concave upward on its entire domain.

\[
\lim_{x \to -\infty} e^x = 0 \quad \text{and} \quad \lim_{x \to \infty} e^x = \infty
\]

\[
e^{a+b} = e^a e^b
\]

\[
e^a e^{-b} = e^{a-b}
\]

Derivative of exponential

Let \( u \) be a differentiable function of \( x \).

1. \[
\frac{d}{dx}[e^x] = e^x
\]

2. \[
\frac{d}{dx}[e^u] = e^u \frac{du}{dx}
\]

Integral of exponential

1. \[
\int e^x \, dx = e^x + C
\]

2. \[
\int e^u \, du = e^u + C
\]
5.5 Bases Other than $e$

**Definition of Exponential Function to Base $a$**

If $a$ is a positive real number ($a \neq 1$) and $x$ is any real number, then the **exponential function to the base $a$** is denoted by $a^x$ and is defined by

$$a^x = e^{(\ln a)x}.$$  

If $a = 1$, then $y = 1^x = 1$ is a constant function.

**Definition of Logarithmic Function to Base $a$**

If $a$ is a positive real number ($a \neq 1$) and $x$ is any positive real number, then the **logarithmic function to the base $a$** is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$  

**Properties of Inverse Functions**

1. $y = a^x$ if and only if $x = \log_a y$
2. $a^{\log_a x} = x$, for $x > 0$
3. $\log_a a^x = x$, for all $x$

**Derivatives**

Let $a$ be a positive real number ($a \neq 1$) and let $u$ be a differentiable function of $x$.

1. $\frac{d}{dx} [a^x] = (\ln a)a^x$
2. $\frac{d}{dx} [a^u] = (\ln a)a^u \frac{du}{dx}$
3. $\frac{d}{dx} [\log_a x] = \frac{1}{(\ln a)x}$
4. $\frac{d}{dx} [\log_a u] = \frac{1}{(\ln a)u} \frac{du}{dx}$

**Integral**

$$\int a^x \, dx = \left( \frac{1}{\ln a} \right) a^x + C$$  

$$\int a^u \, du = \left( \frac{1}{\ln a} \right) a^u + C$$

**Limits involving $e$**

$$\lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = \lim_{x \to \infty} \left( \frac{x + 1}{x} \right)^x = e$$

**Compound Interest:**

- $P =$ amount of deposit
- $T =$ number of years
- $A =$ balance after $t$ years
- $R =$ annual interest rate (decimal form)
- $N =$ number of compoundings per year

- Compounded $n$ times per year:  
  \[ A = P \left( 1 + \frac{r}{n} \right)^{nt} \]

- Compounded continuously:  
  \[ A = Pe^{rt} \]
5.6 Inverse Trigonometric Functions: Differentiation

Definition of Inverse Trig Functions

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \arcsin x$ iff $\sin y = x$</td>
<td>$-1 \leq x \leq 1$</td>
<td>$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$</td>
</tr>
<tr>
<td>$y = \arccos x$ iff $\cos y = x$</td>
<td>$-1 \leq x \leq 1$</td>
<td>$0 \leq y \leq \pi$</td>
</tr>
<tr>
<td>$y = \arctan x$ iff $\tan y = x$</td>
<td>$-\infty &lt; x &lt; \infty$</td>
<td>$-\frac{\pi}{2} &lt; y &lt; \frac{\pi}{2}$</td>
</tr>
<tr>
<td>$y = \arccot x$ iff $\cot y = x$</td>
<td>$-\infty &lt; x &lt; \infty$</td>
<td>$0 &lt; y &lt; \pi$</td>
</tr>
<tr>
<td>$y = \text{arcsec} x$ iff $\sec y = x$</td>
<td>$</td>
<td>x</td>
</tr>
<tr>
<td>$y = \text{arccosec} x$ iff $\csc y = x$</td>
<td>$</td>
<td>x</td>
</tr>
</tbody>
</table>
Properties of Inverse Trig Functions

If $-1 \leq x \leq 1$ and $-\pi/2 \leq y \leq \pi/2$, then
\[ \sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y. \]
If $-\pi/2 < y < \pi/2$, then
\[ \tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y. \]
If $|x| \geq 1$ and $0 \leq y < \pi/2$ or $\pi/2 < y \leq \pi$, then
\[ \sec(\text{arcsec } x) = x \quad \text{and} \quad \text{arcsec}(\sec y) = y. \]

Derivatives of Inverse Trig Functions

\[ \frac{d}{dx} [\arcsin u] = \frac{u'}{\sqrt{1 - u^2}} \quad \frac{d}{dx} [\arccos u] = -\frac{u'}{\sqrt{1 - u^2}} \]
\[ \frac{d}{dx} [\arctan u] = \frac{u'}{1 + u^2} \quad \frac{d}{dx} [\arccot u] = -\frac{u'}{1 + u^2} \]
\[ \frac{d}{dx} [\text{arcsec } u] = \frac{u'}{|u|\sqrt{u^2 - 1}} \quad \frac{d}{dx} [\text{arsec } u] = -\frac{u'}{|u|\sqrt{u^2 - 1}} \]
5.7 Inverse Trig Functions: Integration

Integrals involving inverse trigonometric functions

Let \( u \) be a differentiable function of \( x \), and let \( a > 0 \).

1. \[ \int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C \]

2. \[ \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C \]

3. \[ \int \frac{du}{u \sqrt{u^2 - a^2}} = \frac{1}{a} \text{arcsec} \frac{|u|}{a} + C \]

Techniques needed: